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# Firms' symmetry and sustainability of collusion in a Hotelling duopoly

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## Abstract

We use a differentiated duopoly a la Hotelling to assess the impact of firms' symmetry on the sustainability of a tacit collusive agreement. We obtain that the smaller firm has the greater incentive to deviate and that symmetry helps collusion for any possible differentiation degree.

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## 1. Introduction

Using a model with undifferentiated firms, Motta (2004, pag. 164-165) states that collusion is easier to sustain when firms have similar market shares: symmetry helps collusion. The intuition is simple: since firms are undifferentiated, punishment profits are zero for both firms and deviation profits are also equal. Only collusion profits are different between firms: the firm with a larger market share has larger collusive profits. It follows that the smaller firm has the greater incentive to deviate from the cartel. Moreover, the more the smaller firm is similar to the larger firm the lower is its incentive to cheat.

Things are less obvious when firms are *differentiated*: both punishment and deviation profits are different between firms, and their magnitude depends both on symmetry and differentiation. The purpose of this paper is to assess the impact of firms' symmetry on the sustainability of a collusive agreement between *differentiated* firms. We obtain the following results: the smaller firm is more likely to deviate than the larger firm, and the sustainability of collusion improves with symmetry for any possible differentiation degree between firms. Therefore, Motta (2004) statement is shown to be valid also for differentiated firms[1].

## 2. The model

Following Hotelling (1929), the differentiated good is represented in the product space $\Gamma$ which is the unit interval $[0,1]$. Consumers are uniformly distributed over the interval. Define with $x \in [0,1]$ the location of each consumer. For a consumer positioned at a given point, the most preferred variety is represented by the point in which he is located. Each consumer consumes no more than 1 unit of the good. Define with $v$ the maximum price that a consumer is willing to pay for buying his preferred variety.

There are two firms, $A$ and $B$. Fixed and marginal costs are zero. Firm $A$ is located at $a$, while firm $B$ is located at $b$. Without loss of generality, we assume $0 \le a < b \le 1$. Define $k \equiv b - a$, with $k \in (0,b]$, as the differentiation degree between the two firms: the higher is $k$ (i.e. the more the firms are distant on the segment), the more the firms are differentiated[2].

Define with $p_J$ the price charged by firm $J = A, B$. The utility of a consumer depends on $v$, on the price set by the firm from which he buys, and on the distance between his most preferred variety and the variety produced by the firm. Following D'Aspremont *et al.* (1979), we assume quadratic disutility costs. The utility of a consumer located at $x$ when he buys from firm $A$ is given by:

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[1] Of course, the market share of each firm is only one of the possible dimensions of symmetry/asymmetry among firms. Compte, Jenny and Rey (2002) consider symmetry in capacities, and they show that the firm with the largest capacity has the highest incentive to deviate, and that more symmetry in capacity distribution helps collusion. Kuhn and Motta (1999) consider symmetry in product range: the smaller is the firm (i.e. the lower is the number of product varieties sold) the higher is the temptation to deviate, and the more the firms are similar the more collusion is easy to sustain.

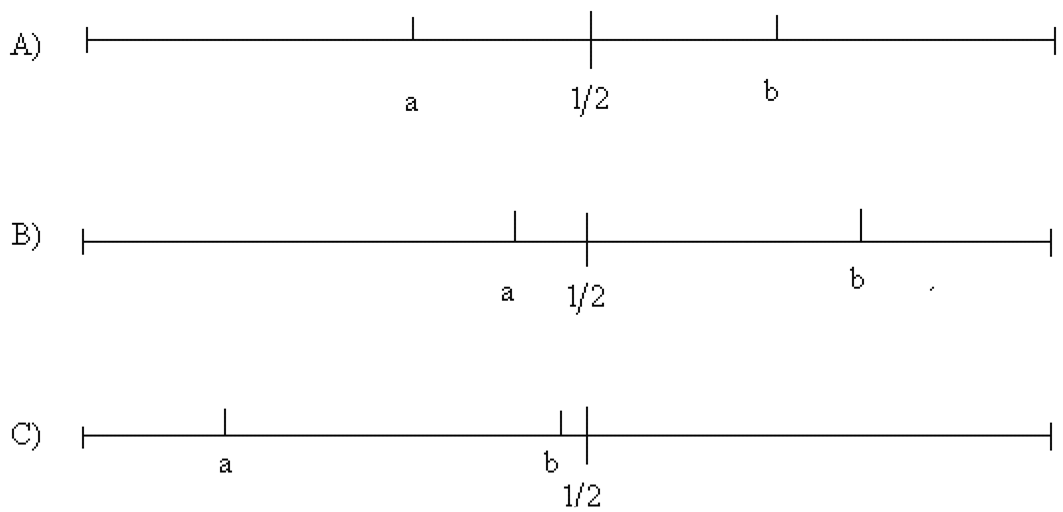
[2] An alternative interpretation of the differentiation degree between the firms within the Hotelling model considers the transportation cost parameter. Here we follow the interpretation adopted, among the others, by Chang (1991, 1992), Friedman and Thisse (1993) and Hackner (1995).

$u_A^x = v - p_A - t(x-a)^2 = v - p_A - t(x-b+k)^2$, while his utility when he buys from firm $B$ is given by: $u_B^x = v - p_B - t(x-b)^2$. We make the following assumption:

Assumption 1: *there is full market coverage*[3].

As in Shaffer and Zhang (2002), we define firms' symmetry as a situation in which, all else equal, the share of the consumers that prefer firm *A* to firm *B* is equal to the share of the consumers that prefer firm *B* to firm *A*. This occurs when $a+b=1$ (Picture 1.A), which implies $b=(k+1)/2$. When firms are asymmetric, there are two possibilities. One possibility is that, all else equal, more consumers prefer firm *A* to firm *B*. This occurs when $a+b>1$ (Picture 1.B), which implies: $b>(k+1)/2$. The other possibility is that, all else equal, more consumers prefer firm *B* to firm *A*. This occurs when $a+b<1$ (Picture 1.C), which implies: $b<(k+1)/2$. In the rest of the article we consider only the case in which firm *B* is "larger" than firm *A*. Therefore, $b \in [k,(k+1)/2)$[4]. Parameter *b* measures the firms' symmetry. The lower is *b*, the more are the consumers that, all else equal, prefer firm *B* to firm *A*. When $b \to 0$, all consumers, all else equal, prefer firm *B* to firm *A*. Instead, when $b \to (k+1)/2$, the share of the consumers preferring firm *A* to firm *B* is equal to the share of the consumers preferring firm *B* to firm *A*. Therefore, the higher is *b* the higher is symmetry[5].

Picture 1




Suppose that firms interact repeatedly in an infinite horizon setting. In supporting collusion, the firms are assumed to use the *grim trigger* strategy of Friedman (1971)[6].

[3] Full market coverage occurs when the reservation price is sufficiently high with respect to transportation cost. It can be shown that $v \geq 4t$ is a sufficient condition for assumption 1 to hold. Details are available from the author upon request.

[4] Note that the assumptions on *b* and *k* can be written even in a compact way: $0<k \leq b<(k+1)/2 \leq 1$.

[5] Clearly, when firm *A* is the larger firm (that is, when $0<(k+1)/2<b \leq 1$ holds) the higher is *b* the lower is symmetry.

[6] The *grim trigger* strategy implies that firms start by charging the collusive price. The firms continue to set the collusive price until one firm has deviated from the collusive agreement in the previous period. If a

Define $\Pi_J^C$, $\Pi_J^D$ and $\Pi_J^N$, with $J = A, B$, respectively as the one-shot collusive, deviation and punishment (or Nash) profits of each firm. The market discount factor, $\delta$, is exogenous and common for each firm. It is well known that collusion is sustainable as a sub-game perfect equilibrium if and only if: $\delta \geq \delta^* = \max[\delta_A, \delta_B]$, where $\delta_J \equiv (\Pi_J^D - \Pi_J^C)/(\Pi_J^D - \Pi_J^N)$, with $J = A, B$. Therefore, $\delta^*$ measures the cartel sustainability: the higher is $\delta^*$ the smaller is the set of market discount factors supporting collusion (i.e. collusion is *less* sustainable), and vice-versa.

## 3. Sustainability of collusion

*The punishment stage*. The punishment price is the Nash-equilibrium price of the one-shot base game. Define with $p_A^N$ and $p_B^N$ respectively the price set by firm *A* and by firm *B*. Define with $\bar{x}$ the indifferent consumer (i.e. the consumer that receives the same utility buying from firm *A* or from firm *B*). By equating $u_A^x$ and $u_B^x$ and solving for *x*, it follows: $\bar{x} = (p_B^N - p_A^N)/2tk + (2b-k)/2$. Since consumers are uniformly distributed, the demand of firm *A* is $\bar{x}$, while the demand of firm *B* is $1-\bar{x}$. The profit function of firm *A* is therefore $p_A^N \bar{x}$, while the profit function of firm *B* is $p_B^N(1-\bar{x})$. Each firm maximizes its profits taking the rival's price as given. Straightforward calculations yield the following Nash-equilibrium prices: $p_A^N{*} = tk(2+2b-k)/3$ and $p_B^N{*} = tk(4-2b+k)/3$. By substituting the equilibrium prices in the profit functions, we obtain the punishment profits:

$$\Pi_A^N = tk(2+2b-k)^2/18 \tag{1}$$
$$\Pi_B^N = tk(4-2b+k)^2/18 \tag{2}$$

Before proceeding, note that firm *A* (*B*) has lower (higher) profits. Moreover, punishment profits of firm *A* (*B*) increase (decrease) with *b*. Therefore, the higher is symmetry the lower is the difference between the punishment profits.

*The collusive stage*. It is not obvious how to define collusive pricing in an asymmetric game. Here we follow Hackner (1994) and assume that firms collude on the uniform price which maximizes joint profits[7]. Define with $\hat{x}$ the consumer paying the highest transportation costs. It turns out that[8]:

<u>Lemma 1</u>: $\hat{x} = \begin{cases} 1 & \forall k \in (0,1/2] \ \ and \ \ \forall k \in (1/2, 2/3] \cap b \leq 1-k/2 \\ b-k/2 & \forall k \in (2/3,1] \ \ and \ \ \forall k \in (1/2, 2/3] \cap b \geq 1-k/2 \end{cases}$

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firm deviates at time *t*, from $t+1$ onward both firms play the equilibrium price emerging in the non-cooperative constituent game.

[7] The main attractiveness of this collusive agreement is its simplicity: if one accepts the idea that colluding firms cannot enforce very complex collusive agreement, the collusive contract we assume is rational. For other possible "ad hoc" collusive contracts between asymmetric firms see for instance Friedman and Thisse (1993).

[8] Except if otherwise mentioned, proofs are relegated to the appendix.

Define $S^1 \equiv \{(b,k)|\hat{x}=1\}$ and $S^2 \equiv \{(b,k)|\hat{x}=b-k/2\}$. The collusive price results from the indifference condition: $v-p^C-t(\hat{x}-b)^2=0$. Solving with respect to $p^C$ and using Lemma 1, it follows:

$$p^C=\begin{cases} v-t(1-b)^2 & \forall(b,k)\in S^1 \\ v-tk^2/4 & \forall(b,k)\in S^2 \end{cases} \tag{3}$$

Let define $\breve{x}$ as the indifferent consumer when the two firms set $p^C$. By equating $u_A^x$ and $u_B^x$ and solving for *x*, we get: $\breve{x}=b-k/2$. Given the uniform distribution assumption, the demand of *A* is $\breve{x}$, while the demand of *B* is $1-\breve{x}$. Therefore, the collusive profits of each firm are:

$$\Pi_A^C=\begin{cases} [v-t(1-b)^2](b-k/2) & \forall(b,k)\in S^1 \\ (v-tk^2/4)(b-k/2) & \forall(b,k)\in S^2 \end{cases} \tag{4}$$

$$\Pi_B^C=\begin{cases} [v-t(1-b)^2](1-b+k/2) & \forall(b,k)\in S^1 \\ (v-tk^2/4)(1-b+k/2) & \forall(b,k)\in S^2 \end{cases} \tag{5}$$

Note that firm *A* obtains lower collusive profits than firm *B*. Moreover, collusive profits of firm *A* (firm *B*) are increasing (decreasing) with *b*[9].

*The deviation stage*. The deviating firm lowers its price in order to steal the rival's consumers. Define $p_J^D$ as the deviation price of firm $J=A,B$. Suppose for the moment that the deviating firm serves the whole market. Therefore, the deviating firm lowers its price until the consumer disliking its variety the most is indifferent between the firms. If the cheating firm is *A*, it sets the highest price which makes the consumer located at 1 indifferent between buying from it or from firm *B*. The indifference condition is therefore:

$$v-p_A^D-t(1-b+k)^2=v-p^C-t(1-b)^2 \tag{6}$$

Substituting (3) into (6) and solving for $p_A^D$ we get:

$$p_A^D=\Pi_A^D=\begin{cases} v-t(1+b^2+k^2-2b+2k-2bk) & \forall(b,k)\in S^1 \\ v-tk(2-2b+5k/4) & \forall(b,k)\in S^2 \end{cases} \tag{7}$$

If the cheating firm is *B*, it sets the highest price which makes the consumer located at 0 indifferent between buying from it or from firm *A*. The indifference condition is:

$$v-p^C-t(b-k)^2=v-p_B^D-tb^2 \tag{8}$$

[9] It can be easily verified that both firms benefit from the collusive agreement, since collusive profits are higher than the Nash profits.

Substituting (3) into (8) and solving for $p_B^D$ we get:

$$p_B^D = \Pi_B^D = \begin{cases} v - t(1-2b-k^2+2bk+b^2) & \forall (b,k) \in S^1 \\ v - tk(2b-3k/4) & \forall (b,k) \in S^2 \end{cases} \tag{9}$$

Differentiation of the deviation profits of firm *A* with respect to *b* shows that deviation profits increase with symmetry. The opposite is true for firm *B*. Moreover, firm *A* has lower deviation profits than firm *B* (details are omitted).
The following lemma states that it is never optimal for the cheating firm to steal only a fraction of the competitor's customers.

Lemma 2: *the deviating firm serves the whole market.*

*The critical discount factor*. Inserting (1), (4) and (7) into $\delta_A$ and (2), (5) and (9) into $\delta_B$, after some algebra we get:

$$\delta_A = \begin{cases} \dfrac{9(-2+2b-k)[v-t(1-2b+b^2+2k)]}{t[18+40k+14k^2+k^3+2b^2(9+2k)-4b(9+7k+k^2)]-18v} & \forall (b,k) \in S^1 \\ \dfrac{9(2-2b+k)(8tk+tk^2-4v)}{2[8tk(10-7b+b^2)+tk^2(37-8b)+2tk^3-36v]} & \forall (b,k) \in S^2 \end{cases}$$

$$\delta_B = \begin{cases} \dfrac{9(2b-k)[t(1-2b+b^2+2k)-v]}{t[18+16k-10k^2+k^3+2b^2(9+2k)-4b(9-5k+5k^2)]-18v} & \forall (b,k) \in S^1 \\ \dfrac{9(2b-k)(8tk+tk^2-4v)}{2[8tk(4+5b+b^2)-tk^2(11+8b)+2tk^3-36v]} & \forall (b,k) \in S^2 \end{cases}$$

We have that:

Proposition 1: *$\delta_A > \delta_B$ and $\delta_A$ is continuous and decreasing in b*[10].

Therefore, the critical discount factor coincides with the one of the smaller firm and symmetry helps collusion. These results extend Motta (2004) analysis to the case of differentiated firms: even if punishment profits and deviation profits are increasing with symmetry, the increase of the collusive profits induced by higher similarity among firms is sufficient to lower the critical discount factor, for any possible differentiation degree.

[10] The proof of the first part of proposition 1 is simply a comparison between $\delta_A$ and $\delta_B$. In order to prove that $\delta_A$ is a continuous function is sufficient to substitute $b = 1 - k/2$ and verify that two parts of the equation coincide. In order to verify that $\delta_A$ is decreasing in *b* it is sufficient to verify that the derivative with respect to *b* is always negative. All calculations are made in Mathematica and available on request from the author.

## Appendix

*Lemma 1*. There are three candidates for being the consumer which pays the highest transportation costs: $x = 0$ (the most to the left consumer), $x = 1$ (the most to the right consumer) and $x = b - k/2$ (the consumer located halfway from the two firms). First, note that the farthest consumer can never be the consumer located at 0. Such consumer buys from firm *A* and pays transportation costs equal to $t(b-k)^2$, while the consumer located at 1 and buying from firm *B* pays $t(b-1)^2$, which is always larger than $t(b-k)^2$. It remains to compare the transportation costs of the consumer located at $b - k/2$, $tk^2/4$, with the transportation costs of the consumer located at 1, $t(b-1)^2$. Therefore, for $x = b - k/2$ to be the consumer suffering the higher disutility the following inequality must by verified: $tk^2/4 \geq t(b-1)^2$. The right-hand side is decreasing in *b*, and has its maximum in $b = k$. Substituting and solving with respect to *k*, it is immediate to note that the inequality is always verified when $k > 2/3$. The right-hand side has its minimum in $b = (k+1)/2$. Substituting and solving with respect to *k*, it is easy to note that the inequality is never verified when $k \leq 1/2$. For intermediate values of *k*, we solve the inequality with respect to *b*, and we obtain that it is verified when $1 - k/2 \leq b \leq 1 + k/2$ (clearly, the second inequality is always satisfied).

*Lemma 2*. Consider first firm *A*. The most at the right consumer served by firm *A* comes from the indifference condition: $v - p_A^{D,f} - t(x-b+k)^2 = v - p^C - t(x-b)^2$, where $p_A^{D,f}$ indicates that the deviating firm serves only a fraction of the market. Solving with respect to *x* it follows: $x^* = (p^C - p_A^{D,f})/2tk + (2b-k)/2$. The profit function of *A* is therefore $p_A^{D,f} x^*$. Maximizing the profit function with respect to the price, we obtain: $p_A^{D,f*} = p^C/2 + tk(2b-k)/2$. Substituting equation (3) into $p_A^{D,f*}$ we obtain:

$$p_A^{D,f*} = \begin{cases} [v - t(1 + b^2 + k^2 - 2b - 2bk)]/2 & \forall (b,k) \in S^1 \\ [4v - tk(5k - 8b)]/8 & \forall (b,k) \in S^2 \end{cases} \tag{10}$$

If $p_A^{D,f*} \leq p_A^D$, it is better for firm *A* to serve the whole market. We have:

$$p_A^{D,f*} - p_A^D = \begin{cases} [t(1 + b^2 + 4k + k^2 - 2b - 2bk) - v]/2 & \forall (b,k) \in S^1 \\ [tk(5k - 8b + 16) - 4v]/8 & \forall (b,k) \in S^2 \end{cases} \tag{11}$$

Clearly, $p_A^{D,f*} - p_A^D$ is decreasing in *v*. By substituting $v = 4t$ (see footnote 3) into equation (11) we get:

$$p_A^{D,f*} - p_A^D = \begin{cases} t[b(b-2) + k(k-2b) + (4k-3)]/2 & \forall (b,k) \in S^1 \\ t[16(k-1) + k(5k-8b)]/8 & \forall (b,k) \in S^2 \end{cases} \tag{12}$$

From the assumptions on $b$ and $k$ it follows immediately that $b(b-2)<0$, $k(k-2b)<0$, $k-1<0$ and $k(5k-8b)<0$. Moreover, $(b,k)\in S^1$ implies $k\leq 2/3$, which in turn implies $4k-3<0$. It follows that equation (12) is negative: therefore the deviating firm serves the whole market.

Suppose now that the deviating firm is *B*. The most at the left consumer served by firm *B* is the solution with respect to $x$ of the following indifference condition: $v-p^C-t(x-b+k)^2=v-p_B^{D,f}-t(x-b)^2$. Then, $x^*=(p_B^{D,f}-p^C)/2tk+(2b-k)/2$. The profit function of firm *B* is: $p_B^{D,f}(1-x^*)$. Maximizing with respect to the price, we get: $p_B^{D,f}*=tk(2-2b+k)/2+p^C/2$. Substituting (3) into $p_B^{D,f}*$, we get:

$$p_B^{D,f}*=\begin{cases}[v-t(-1-b^2+k^2+2b-2bk+2k)]/2 & \forall(b,k)\in S^1\\ [4v-tk(8b-8-3k)]/8 & \forall(b,k)\in S^2\end{cases} \tag{13}$$

If $p_B^{D,f}*\leq p_B^D$, it is better for firm *B* to serve the whole market. We have:

$$p_B^{D,f}*-p_B^D=\begin{cases}[t(1+b^2-k^2-2b+2bk+2k)-v]/2 & \forall(b,k)\in S^1\\ [tk(8+8b-3k)-4v]/8 & \forall(b,k)\in S^2\end{cases} \tag{14}$$

Clearly, $p_B^{D,f}*-p_B^D$ is decreasing in *v*. By substituting $v=4t$ into equation (14) we get:

$$p_B^{D,f}*-p_B^D=\begin{cases}t[b(b-2)+(-9-k^2+2bk+2k)]/2 & \forall(b,k)\in S^1\\ t[-3k^2+(8bk+8k-16)]/8 & \forall(b,k)\in S^2\end{cases} \tag{15}$$

From the assumptions on $b$ and $k$ it is immediate to see that all terms in equation (15) are negative. This implies that the deviating firm serves the whole market.

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