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Strict dominance solvability without equilibrium

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Abstract

A game in strategic form is strict dominance solvable if iterative elimination of strictly dominated strategies yields a unique strategy profile (strict dominance solution). Textbook presentations of this material are framed in the context of finite games and it is argued that if a strict dominance solution exists, it must also be the unique Nash equilibrium. We construct a simple counter example demonstrating that strict dominance solutions need not constitute Nash equilibria in infinite games, even if each player has a unique undominated strategy. This conclusion has special pedagogical significance as the sensitivity of results to the finite game context can often be lost on those being introduced to the material for the first time. As an additional pedagogical exercise, we establish that the traditional textbook conclusion extends to settings in which strategy spaces are compact and utility functions are continuous.

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1. Introduction

A standard feature of game theory textbooks, especially those pitched at the graduate level, is a presentation on the iterative elimination of strictly dominated strategies and the implications this process has on rational play. This general discussion begins by noting that rational players can be thought of as being precluded from selecting strictly dominated strategies. This insight is extended when player objectives are assumed to be common knowledge as attention can thus be focused on the residual game in which strictly dominated strategies have been eliminated from each player's strategy set. A game is said to be dominance solvable when the iterative application of this logic to the resultant residual games yields a unique strategy profile. Following the logic outlined above, this "strict dominance solution," when it exists, is argued to be the only outcome that can be pursued by players who are commonly known to be rational. Mas-Colell, Whinston, and Green (1995), page 239, provides a representative quote. "As a general matter, if we are willing to assume that all players are rational *and* that this fact and the players' payoffs are common knowledge (so everybody knows that everybody knows that… everybody is rational), then we do not need to stop after only two iterations. We can eliminate not only strictly dominated strategies and strategies that are strictly dominated after the first deletion of strategies but also strategies that are strictly dominated after this next deletion of strategies, and so on." The standard presentation of iterative strict dominance goes on to conclude that the strict dominance solution is endowed with singular equilibrium features. For instance, Fudenberg and Tirole (1991), page 46, states that, "… this profile is necessarily a Nash equilibrium (indeed, it is the unique Nash equilibrium)." Similar discussions appear in the texts of Gibbons (1992), Moulin (1986), Osborne and Rubinstein (1994), and Vega-Redondo (2003). These discussions are, however, framed either implicitly or explicitly in the context of finite games. We present a straightforward counter example that can be used to highlight the significance that the finite game context has in the fore mentioned conclusion. This example further demonstrates that when there are infinitely many strategies, even if every player has a unique undominated strategy (and thus an iterative appeal to the common knowledge of rationality is unnecessary), this strategy profile need not constitute a Nash equilibrium. As an additional pedagogical exercise, we generalize the standard finite game conclusion and show that if all strategy sets are compact and all utility functions are continuous, strict dominance solutions will necessarily constitute Nash equilibria.

2. Strict Dominance Solvability and a Counter-Example

We begin by formally defining the notion of dominance solvability. Take as given a game in strategic form $G=\{I, (S_i)_{i\in I}, (u_i)_{i\in I}\}$. That is, I denotes the finite set of players, $S_{i'}$ denotes the pure strategy space (which may be infinite) of player $j\in I$, and $u_j: (S_i)_{i\in I}\to\mathbb{R}$ denotes the utility function of player $j\in I$. So as to avoid the technical details that ensue when defining mixed strategies of infinite strategy spaces, we define the notion of strict domination solely in terms of pure strategies. As we argue below, this expository simplification in no way compromises our conclusions. Readers interested in a textbook presentation of mixed strategies in the context of infinite strategy spaces may consult Myerson (1994), section 3.13.

Definition 2.1 Suppose $S_j' \subseteq S_j$ for each $j \in I$. The strategy $s_i \in S_i'$ is strictly dominated for player i given $(S_j')_{j\in I}$ if there exists $s_i' \in S_i'$ such that $u_i(s_i', s_{-i}) > u_i(s_i, s_{-i})$ for all $s_{-i} \in S_{-i}' = (S_j')_{j\neq i}$.

Definition 2.2 Let $S_i^0 = S_i$ for each $i \in I$ and iteratively construct S_i^n for each positive integer $n=1,2,\ldots$, by: $S_i^n = \{s_i \in S_i^{n-1} \mid s_i$ is not strictly dominated for player i given $(S_j^{n-1})_{j\in I}\}$. The game $G = \{I, (S_i)_{i\in I}, (u_i)_{i\in I}\}$ is *strict dominance solvable* if $\bigcap_{n=0}^{\infty} S_i^n$ contains a single strategy for each $i \in I$ and we refer to the strategy profile constituted by these strategies as the *strict dominance solution*.

We now construct a simple counter example demonstrating that a strict dominance solution need not be a Nash equilibrium, even if it can be achieved through a single application of strict dominance elimination. Let the components of the game $G = \{I, (S_i)_{i\in I}, (u_i)_{i\in I}\}$ be defined as follows.

$I = \{1,2\}$,

$S_1 = S_2 = \{x \mid x = \frac{1}{z}$ for some positive integer $z\} \cup \{0\}$,

$u_1(s_1,s_2) = 1 - s_1$ for each $s_1 \in S_1 \backslash \{0\}$ and each $s_2 \in S_2$,
$u_1(0,s_2) = 1$ for each $s_2 \in S_2 \backslash \{0\}$,
$u_1(0,0) = 0$,

$u_2(s_1,s_2) = 1 - s_2$ for each $s_1 \in S_1$ and each $s_2 \in S_2 \backslash \{0\}$,
$u_2(s_1,0) = 1$ for each $s_1 \in S_1 \backslash \{0\}$, and
$u_2(0,0) = 0$.

Note that $u_1(\bullet, s_2)$ is independent of s_2 and strictly decreasing when constrained to the domain $S_1 \backslash \{0\}$. Moreover, $s_1 = 0$ is a better reply to $s_2 \in S_2 \backslash \{0\}$ than is any $s_1 \in S_1 \backslash \{0\}$. Consequently, 0 is the only strategy in S_1 that is not strictly dominated for player 1. Indeed, $s_1 > 0$ is strictly dominated by s_1' whenever $s_1' < s_1$ and $s_1' \neq 0$. Likewise, 0 is the only strategy in S_2 that is not strictly dominated for player 2. It follows that $S_1^1 = S_2^1 = \{0\}$ and the strict dominance solution is realized after only one round of eliminating strictly dominated strategies, where S_1^1 and S_2^1 are as defined in Definition 2.2. Nonetheless, $(s_1, s_2) = (0,0)$ is clearly not a Nash equilibrium in the original game G, as can be seen by noting that $u_1(0,0) = 0$ and $u_1(s_1,0) = 1 - s_1 > 0$ for each $s_1 \in S_1 / \{0\}$.

The utility functions in the example above are discontinuous at 0, suggesting that it is this discontinuity that is responsible for results that differ from the traditional finite game context. A straightforward modification of the example reveals that compactness also plays a critical role. In particular, let us reformulate strategy sets as $S_1 = S_2 = \{x \mid x = n$ for some nonnegative integer $n\}$.

Furthermore, let $u_1(s_1,s_2)= 1-(1/s_1)$ for each $s_1 \in S_1 \backslash \{0\}$ and each $s_2 \in S_2$ and let $u_2(s_1,s_2)= 1-(1/s_2)$ for each $s_1 \in S_1$ and each $s_2 \in S_2 \backslash \{0\}$, while continuing to let $u_1(0,0)=u_2(0,0)=0$, and $u_1(0,s_2)=u_2(s_1,0)=1$ for all $s_1 \in S_1 \backslash \{0\}$ and $s_2 \in S_2 \backslash \{0\}$. Utility functions in this reformulated game are continuous under the usual Euclidean topology, while the strategy spaces now fail to be compact. The strategy profile $(0,0)$ continues to be the strict dominance solution and it also continues to fail to constitute a Nash equilibrium in the underlying game.

At this juncture it is worth further noting that the profile $(s_1, s_2)=(0,0)$ would continue to be a strict dominance solution for G even if Definition 2.1 were formulated so as to allow for domination via mixed strategies. Indeed, this is trivially true given that for each $i=1,2$, $s_i=0$ fails to be strictly dominated for player i by any strategy, mixed or pure.

3. Strict Dominance Nash Solutions

We conclude with a theorem establishing that strict dominance solutions must also be Nash equilibria when consideration is restricted to games in which each player's utility function is continuous and each player's strategy space is compact. As the proof of this result does not depend on whether or not strict domination via mixed strategies is "allowed," we see that once again the results we offer do not depend on our pure strategy formulation of strict domination.

Theorem 3.1 Let $G= \{I, (S_i)_{i \in I}, (u_i)_{i \in I}\}$ be a game such that S_i is compact and u_i is continuous for each $i \in I$, then a strict dominance solution of G, if one exists, must also be a Nash equilibrium.

Proof: Let $G= \{I, (S_i)_{i \in I}, (u_i)_{i \in I}\}$ be a game satisfying the theorem's hypotheses and suppose that there exists a strict dominance solution $(s_i^*)_{i \in N}$ of G that is not a Nash equilibrium, that is, there exists $i \in I$ such that s_i^* is not a best reply to s_{-i}^*. Compactness of S_i and continuity of u_i thus implies there exists $s_i' \in S_i$ such that $u_i(s_i',s_{-i}^*) \geq u_i(s_i,s_{-i}^*)$ for all $s_i \in S_i$ and $u_i(s_i',s_{-i}^*) > u_i(s_i^*,s_{-i}^*)$. Note that $s_i' \neq s_i^*$ and definition of strict dominance solvability implies there exists a nonnegative integer n such that $s_i' \in S_i^n$ and $s_i' \notin S_i^{n+1}$, in turn implying there exists $s_i \in S_i^n$ such that $u_i(s_i,s_{-i}) > u_i(s_i',s_{-i})$ for all $s_{-i} \in S_{-i}^n$. But $s_{-i}^* \in S_{-i}^n$, implying that $u_i(s_i,s_{-i}^*) > u_i(s_i',s_{-i}^*)$ and thus contradicting the assumption that $u_i(s_i',s_{-i}^*) \geq u_i(s_i,s_{-i}^*)$ for all $s_i \in S_i$. QED

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