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A note on fixed and flexible-term contracts

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Abstract

Engel et al. (1997 2001 2007) have proposed flexible-term contracts as a solution to suboptimal renegotiation of public-private partnerships. We show that whenever the uncertainty lasts over more than two periods, flexible-term contracts have a drawback. The expected duration of a flexible-term contract can be higher than the (certain) duration of a fixed-term contract, the difference being directly proportional to the volatility of the project.

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1 Introduction

Public-private partnerships are contractual forms between public procurements and full privatization. Public authorities involve private firms in public projects because they need private funds for large investments in infrastructure. At the same time, full privatization is not desirable because the public authorities need to keep the control of the activities of public service provision, for which market competition does not exist. Under a public-private partnership, the project is private only during the period of the contract. At the end of the contract the ownership of the infrastructure reverts to the public authority. Ceteris paribus, the shorter the contract duration the higher the public benefit of the project.

Engel et *al.* (1997; 2001; 2007) argue that a public-private partnership for building and operating highways must, at optimum, take the form of flexible-term contracts. The contract stipulates the return that the firm has to obtain from the project and lasts until the cumulated cash flow hits this return.

The reason why flexible-term contracts were proposed is that they have some advantages over fixed-term contracts, whenever uncertainty exists. In transportation infrastructure investments, uncertainty has led to frequent renegotiations in bad states of the world, when cash flows were low and so did not allow for cost recovery (Guash, 2004). Under a flexible-term contract the return of the firm is certain (or less risky) and thus renegotiation is avoided, or, at least less likely. Indeed, the firm is allowed to operate over a longer period whenever the cash flow is low, until the cumulated cash flow hits the contracted return. In turn, the duration of the contract is shortened in a good state of the world, in which the contracted return is obtained rapidly by the firm.

Engel et *al.* (1997; 2001; 2007) show the advantage of flexible-term contracts in the context of two-period uncertainty and a risk averse private firm. In this study we show that when the uncertainty lasts over multiple periods, the expected contract duration can be suboptimally high. The scope of this study is to signal that flexible-term contracts have a drawback that should be accounted for in the analysis of the optimal contractual regime. For simplicity of the exposition, we assume that the private partner is risk neutral but further extensions can be made to assume that the firm is risk averse.

The fact that the returns of infrastructure projects are uncertain is obvious. For instance, traffic prevision is subject to significant errors, as many empirical studies have shown[1].

[1] One example is the study of Quinet (2000), who distinguishes three sources of inaccuracy: the model structure, current data and future value of exogenous variables. Small and Winston (1998) argue that uncertainty about future traffic comes on the one side from economic conditions, technology etc (i.e. exogenous factors), which are difficult to forecast accurately and on the other side from the fact that the information is not transferable across time and space. Skamris and Flybjerg (1997) do an empirical comparison between roads and railroads and show that forecast of railroads seem to be technically more problematic than that

Moreover, to see why uncertainty lasts over multiple periods, one could think about the influence of GDP on the demand for transportation. GDP is exogenous to the transportation sector and its dynamics determine permanent and uncertain shocks on the demand[2]. Because GDP is dynamic, the uncertainty about the demand lasts all over the future.

We explain briefly the result. Let us denote $E(T)$ the expectation of the flexible period T, which is the necessary time that allows the firm to cumulate a certain reservation utility u. We show that $E(T)$ is weakly higher than T', the duration of the fixed-term contract, which is necessary for the firm to obtain u in expectation, i.e. to satisfy its participation in the project. $E(T) = T'$ only in the specific case in which the uncertainty does not exist, i.e. in the situation in which flexible-term contracts become unnecessary.

Formally, this difference can be explained as follows. Under flexible-term contracts the cash flow of the firm is upper bounded by some level at which the cumulated cash flow reaches the value u. Indeed, the contract ends when u is obtained. However, in bad states of the world, the cash flow of the firm is not bounded from below. It can go as low as possible while the contract lasts until the cash flow moves up to the upper bound at which u is obtained. Hence an *asymmetry* between up and down shifts of the cash flow of the firm exists. When the contract has a fixed-term, such asymmetry does not exist. The firm takes the risk of down shifts of the cash flow, but it also takes the advantage of any upstream evolution of the cash flow. In expectation, the cumulated market benefit of the firm in an uncertain market is expected to be higher when upstream revenues are not limited. Consequently, this upper boundary leads to the inequality $E(T) \geq T'$.

The remainder is organized as follows. Section 2 presents a discrete time example of the issue with three and four-period uncertainty. Section 3 describes the general model. Section 4 concludes.

2 Discrete time example

We take first an example with three-period uncertainty and no discount factor. The cash flow y_t of the firm is distributed according to a simple random walk. Figure 1 shows the possible states of y_t. The initial state is $y_0 = 1$. y_t shifts up or down with 1 unity and equal probabilities between any two periods from $t = 0$ to $t = 3$.

of roads.

[2] A particular trend of the literature is to investigate the time lag over which demand responds to changes in its determinants, in particular in GDP (see for instance, Goodwin, 1976 and Oxera, 2005).

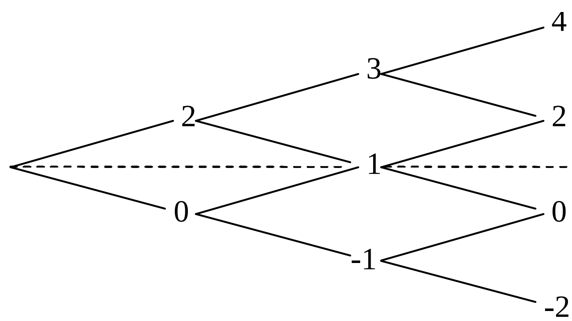

Figure 1: States of y_t under three-period uncertainty

Assume that the firm is risk neutral and has zero reservation utility. The firm makes an investment $I = 2$, so that it participates if it obtains a cumulated cash flow of $u = 2$. The utility u can be offered to the firm in expectation, through a fixed-term contract T', such that $E(\widetilde{u}) = u$. Alternatively, u is offered with certainty, through a flexible-term contract T.

If the fixed-term contract is offered to the firm, then the participation of the firm is ensured if $T' = 2$. In this way the firm obtains an expected return

$$u = E(\widetilde{u}) = E(y_1) + E(y_2) = 1 + 1 = 2$$

Note that the value of y_0 is relevant only for the forecast of future states; the firm starts producing at date 0 and obtains a profit only from period 1. Also, because the firm is risk neutral, the uncertainty about y_t is irrelevant for its reservation utility. The firm obtains in expectation $E(\widetilde{u}) = u$ while the real utility $\widetilde{u}$ can be either 5, 2 or -1. So the firm can obtain a high profit $5 - 2 = 3$ but it can also face losses of -3.

Under a flexible-term contract the contract duration T' is determined as the first passage time at which the cumulated profit of the firm is at least $u = 2$. In our example, $T = 1$ whenever $y_1 = 2$, which occurs with probability $1/2$. With the same probability, $y_1 = 0$, in which case $T \geq 3$. The inequality is strict if $y_1 = 0$ is followed by $y_3 = 0$ or -2. Overall, $T = 1$ with probability $1/2$, $T = 3$ with probability $1/8$ and $T > 3$ with probability $3/8$. Evidently, $E(T) > T'$.

Suppose now that uncertainty lasts between $t = 0$ and $t = 4$, as in the Figure 2 below.

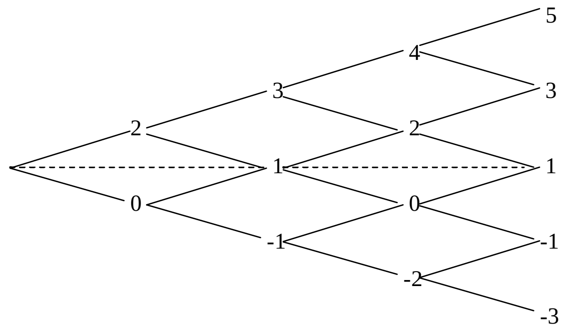

Figure 2: States of y_t under four-period uncertainty

As previously, if $y_1 = 2$ then $T = 1$. If $y_1 = 0$, $y_2 = 1$ and $y_3 = 2$ then $T = 3$. Otherwise, $T > 3$. The novelty with respect to the previous example is that whenever $y_i < 2$, for $i \in \{1, 2, 3\}$, the contract duration T is such that $T > 4$. Evidently, $E(T) > T'$ as before, but at the same time $E(T)$ is higher under four-period uncertainty than under three-period uncertainty. The same reasoning applies when any new period of uncertainty is added to the problem. In the realistic case in which uncertainty lasts infinitely, there is some positive probability that T lasts infinitely.

3 Continuous time model

Assume now that y_t follows a geometric Brownian motion with drift $\alpha > 0$ and volatility $\sigma \geq 0$, such that

$$dy_t = \alpha y_t dt + \sigma y_t dz_t, \tag{1}$$

with z_t a simple Brownian motion.

3.1 Fixed-term contract

We assume for simplicity that the life time of the project is the infinity. The term T' of the contract is chosen in a way that the firm obtains in expectation its reservation utility: $E(\widetilde{u}) = u$. The payoff of the firm is

$$V_{fi}(y_0) = \mathrm{E}_{y_0}\left(\int_0^\infty y_t e^{-rt} dt - \int_{T'}^\infty y_t e^{-rt} dt\right), \tag{2}$$

with E_{y_0} the expectation operator, given the current state y_0.

The first term of $V_{fi}(y_0)$ is the expected discounted cash flow of the project. Because the contract ends at time T', the discounted cash flow of all periods $t > T'$ is substrated from $V(y_0)$. The contracting term T' is fixed, while the exact value y_{fi} at which the contract ends is unknown. To find its expected value, we can write from (1) the expression

$$E_{y_0}\left(y_t\right) = y_0 e^{\alpha t}, \ \forall \ t \geq 0, \tag{3}$$

so that for $t = T'$, $E_{y_0}\left(y_{fi}\right) = y_0 e^{\alpha T'}$. From (2) and (3) we can write

$$V_{fi}\left(y_0\right) = \frac{y_0}{r-\alpha} - e^{-(r-\alpha)T'} \frac{y_0}{r-\alpha}$$

The contract duration T' must solve the equality $V\left(y_0\right) = u$, which is equivalent to

$$T' = \ln \left(1 - u\left(\frac{y_0}{r-\alpha}\right)^{-1}\right)^{-\frac{1}{r-\alpha}} \tag{4}$$

(4) shows that T' is independent of the volatility parameter σ. The firm is given u in expectation while the risk of the project is borne by the firm.

3.2 Flexible-term contract

We define $\tau = \inf\left\{t \geq 0 \ s.t. \ y_t = y_{fl}\right\}$ the termination period of the flexible-term contract (previously T). τ is the first time at which the hitting value y_{fl} is reached, i.e. the time at which the market profit y_{fl} is such that the firm obtains its reservation utility u. The Appendix shows the standard calculation for the derivation of the payoff of the firm. For any $y < y_{fl}$, the value of the project is

$$\begin{aligned} V_{fl}\left(y\right) &= E_y\left(\int_0^{\infty} y_t e^{-rt} dt\right) - \left(\frac{y}{y_{fl}}\right)^{\beta} E_{y_{fl}} \int_0^{\infty} y_t e^{-rt} dt \\ &= \frac{y}{r-\alpha} - \left(\frac{y}{y_{fl}}\right)^{\beta} \frac{y_{fl}}{r-\alpha}, \end{aligned} \tag{5}$$

with β is the positive root of the quadratic

$$Q\left(x\right) = \beta\left(\beta - 1\right)\frac{\sigma^2}{2} + \alpha\beta - r \tag{6}$$

The first term in the expression of $V\left(y\right)$ is the expected discounted cash flow that the firm would obtain if the contract lasted infinitely. The second term substrates from this value the discounted cash flow that corresponds to the periods beyond the contract. The contract ends when some value y_{fl} is hited by the stochastic variable for the first time. The expression $\left(\frac{y}{y_{fl}}\right)^{\beta}$ is the discounting operator in the space of realizations of the stochastic variable, so that

$$\left(\frac{y}{y_{fl}}\right)^{\beta} = E\left(e^{-r\tau}\right) \tag{7}$$

The following equation must be satisfied: $V(y_0) = u$. This equation states the meaning of the flexible-term contract: the firm obtains a fixed reservation utility u, from cumulating market profits during an uncertain time interval. Hence we can derive the value y_{fl} that triggers the end of the contract, as follows

$$y_{fl} = y_0\left(1 - u(r-\alpha)\,y_0^{-1}\right)^{\frac{1}{1-\beta}} \tag{8}$$

We move now to the comparison between $T(y_0)$ and T', where $T(y_0)$ is the expected duration of the flexible-term contract, such that

$$T(y_0) = E(\tau)$$

We need in this sense to analyze the discount factor (7). The right hand side of (7) is a convex function of τ. It follows by Jensen inequality that $E\left(e^{-r\tau}\right) \geq e^{-rT(y_0)}$. We can then write the following inequality

$$e^{-rT(y_0)} \leq \left(\frac{y_0}{y_{fl}}\right)^{\beta} \Leftrightarrow T(y_0) \geq \frac{\beta}{r}\ln\left(\frac{y_{fl}}{y_0}\right) \tag{9}$$

From (8) and (9) we can write

$$T(y_0) \geq \ln\left(1 - u\frac{r-\alpha}{y_0}\right)^{-\frac{\beta}{\beta-1}\frac{1}{r}} \tag{10}$$

With this result we can show the following.

Theorem 1. *The expected duration $T(y_0)$ of the flexible-term contract and the duration T' of the fixed-term contract are such that $T(y_0) \geq T'$. $T(y_0) = T'$ if and only if $\sigma = 0$.*

Proof. Show first $T(y_0) \geq T'$. We calculate

$$\ln\left(1 - u\frac{r-\alpha}{y_0}\right)^{-\frac{\beta}{\beta-1}\frac{1}{r}} \geq \ln\left(1 - u\left(\frac{y_0}{r-\alpha}\right)^{-1}\right)^{-\frac{1}{r-\alpha}} \Leftrightarrow$$

$$\ln\left(1 - u\frac{r-\alpha}{y_0}\right)^{\frac{1}{r-\alpha}-\frac{1}{r}\frac{\beta}{\beta-1}} \geq 1$$

From (4), (10) and the above inequality, $T(y_0) \geq T'$.

Show now $T(y_0) = T' \Leftrightarrow \sigma = 0$. From $T(y_0) = E(\tau)$ and $\sigma = 0$ we deduce $T(y_0) = \tau$. Also, from $\sigma = 0$ and (6), we can write $\frac{\beta}{\beta-1} = \frac{r}{r-\alpha}$. Using (4), (10) together with $T(y_0) = \tau$ and $\frac{\beta}{\beta-1} = \frac{r}{r-\alpha}$, we calculate

$$\begin{aligned} \sigma &= 0 \Leftrightarrow E\left(e^{-r\tau}\right) = e^{-rT(y_0)} \Leftrightarrow T(y_0) = \ln\left(1 - u\frac{r-\alpha}{y_0}\right)^{-\frac{\beta}{\beta-1}\frac{1}{r}} \\ &\Leftrightarrow T(y_0) = T' \end{aligned}$$

$\square$

The Appendix shows that the expression of $T(y_0)$ is

$$T(y_0) = \begin{cases} \frac{1}{\alpha - \frac{1}{2}\sigma^2} \ln\left(1 - u(r-\alpha)\, y_0^{-1}\right)^{\frac{1}{1-\beta}}, & \text{if } \alpha > \frac{1}{2}\sigma^2 \\ \infty, & \text{if } \alpha \leq \frac{1}{2}\sigma^2 \end{cases} \tag{11}$$

Using (11), we show in Figure 1 how the expected contracting period of the flexible-term contract diverge from the duration of the fixed-term contract.

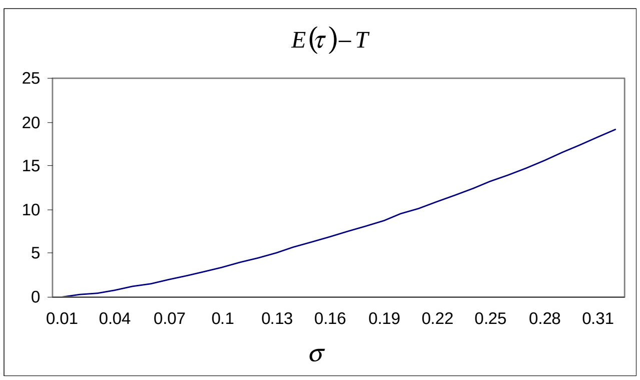

Figure 3: $E(\tau) - T$ as a function of σ

Dixit and Pindyck (1994) show that upper bound barriers often occur in economics because of equilibrating mechanisms of the market. In a flexible-term contract, it is not the market that puts a frontier on the value of the project but the contract, which specifies that the firm is not allowed to obtain returns that are over the threshold u. In a fixed-term contract, such barrier does not exist. This is why for the same reservation utility u that must be given to the firm, $T(y_0) \geq T'$.

4 Conclusion

We have shown that the expected duration of the flexible-term contract is higher than the duration of the fixed-term contract. This result holds under the assumption that uncertainty lasts over more than two periods and the private firm is risk neutral.

A natural continuation of this study would be to investigate the optimal contractual regime. In particular, when the firm is able to renegotiate the contract in bad state of the world, as is usually the case in public-private partnerships, the cash flow of the firm is bounded from below, both under fixed and flexible-term contract. One could investigate which type of contract is second best optimal, given that the risk born by the firm is limited by renegotiation.

Furthermore, the literature on public-private partnerships has shown that there is a link between contract duration and the incentives of the firm to underinvest. Ceteris paribus the firm invests more if the duration is higher because cash flows of each period are directly proportional to the amount of their investment (see Iossa and Martimort, 2008). One can infer from our result that the incentives to underinvest are not the same under fixed and flexible-term contracts since they have distinct expected duration. It would be then interesting to compare the incentives to invest under the two contractual regimes.

The analysis of both renegotiation and incentives to invest under fixed and flexible-term contracts is left for further research.

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Appendix

Find $V_{fl}(y)$

From (1) and applying Ito's lemma,

$$\mathrm{E}_y (dV_{fl}) = \alpha y V'_{fl}(y)\, dt + \frac{1}{2}\sigma^2 y^2 V''_{fl}(y)\, dt \tag{12}$$

The value of the project for any $y_t = y$ and $y < y_{fl}$ is

$$V_{fl}(y) = ydt + \mathrm{E}_y (dV_{fl})\, e^{-rdt} \tag{13}$$

From (12) and (13), we find the differential equation

$$\alpha y V'_{fl}(y) + \frac{1}{2}\sigma^2 y^2 V''_{fl}(y) - rV_{fl}(y) + y = 0 \tag{14}$$

The initial condition is $V_{fl}(0) = 0$. From (14) and the initial condition we can rewrite the Bellman equation (13) as

$$V_{fl}(y) = \frac{y}{r-\alpha} + ay^{\beta} \tag{15}$$

At $y = y_{fl}$, we can write $V_{fl}(y_{fl}) = 0$. Indeed, for any $y \geq y_{fl}$ the firm does not obtain any cash flow because the contract ends. From the above expression of $V_{fl}(y)$ and from $V_{fl}(y_{fl}) = 0$ one deduces

$$a = -\left(\frac{1}{y_{fl}}\right)^{\beta} \frac{y_{fl}}{r-\alpha}$$

By replacing it in (15) we find the expression (5) in the main text.

Find $T(y_0)$

The demonstration follows closely Dixit (1993). We start from a random walk x, whose expected intertemporal shift is

$$E[\Delta x] = (p-q)\,\Delta h,$$

where p is the probability of up move and $q = 1-p$ the probability of down move. Δx takes values $\pm\Delta h$, where $\Delta h > 0$. Assume that x_t can take values between a and b, $a < b$. We denote i a natural number between $-n$ and 0, where $n > 0$. $i = -n$ corresponds to $x = a$ and $i = 0$ corresponds to $x = b$. The general formula for the states of the random walk as function of the index i is $x = b + i * \Delta h$. Denote also T_i the expected time of a future event, provided that the initial state is i. T_i satisfies the following difference equation

$$T_i = \Delta t + qT_{i-1} + pT_{i+1}. \tag{16}$$

The expected time of any future event is the sum of the next step Δt and the expected remaining steps. With probability q the value y is reached from below and with probability p is reached from above. This is a inhomogenious difference equation. The homogeneous part of (16) is similar to

$$M_i = qM_{i-1} + pM_{i+1}.$$

We try a solution of the form $M_i = \beta^i$. Using $p + q = 1$ the equation is verified for $\beta = 1$ and $\beta = (q/p)$. These two solutions are independent if $q \neq p$, so that the general solution of the homogenous equation is

$$M_i = A + B\,(q/p)^i\,. \tag{17}$$

The constant of the difference equation leads us to guess as particular solution

$$T_i = E * i + F \tag{18}$$

If we replace this in the original equation we find that (16) is verified by whatever real value F and by the value of E that solves

$$E = -\frac{\Delta t}{p-q}. \tag{19}$$

Using (17) (18) and (19) in (16), we can write the general solution of the difference equation as

$$T_i = -i\frac{\Delta t}{p-q} + A' + B\,(q/p)^i$$

The constants A' and B are found from the boundary conditions. We use here that $T_{-n} = T_0 = 0$, i.e., the extreme points n and 0 are absorbing barriers for the random walk process. We find

$$T_i = \frac{\Delta t}{p-q}\left(-n\frac{1-(q/p)^i}{1-(q/p)^{-n}} - i\right).$$

We are interested in the situation in which the state x is bounded only from above, at b. Therefore $a = -\infty$, which implies $n = \infty$. Calculating the limit of the above expression, we can write

$$T_i = \begin{cases} -i\frac{\Delta t}{p-q}, \ if\ p > q \\ \infty, \ if\ p < q \end{cases}. \tag{20}$$

Remember that $i \leq 0$, so that T_i is non-negative, which makes sense.

We use this for finding the time of a future event of an arithmetic Brownian motion. Define $\Delta h = \sigma\sqrt{\Delta t}$. Also, $p - q = \frac{\mu}{\sigma^2}\Delta h$. With these definitions, Dixit (1993) show that as $\Delta t \to 0$ Δx becomes an arithmetic Brownian motion of mean μdt and variance σdt. Replace these definitions in (20), together with $i = \frac{x-b}{\Delta h}$. (20) becomes

$$T(x) = \begin{cases} \frac{b-x}{\mu}, \text{ if } \mu \geq 0, \\ \infty, \text{ if } \mu < 0 \end{cases}. \tag{21}$$

Move now from arithmetic to geometric Brownian motion, as in the main text. Use $x = \ln(y_t)$, where y_t follows the geometric Brownian motion defined in the main text. Then, by Ito's lemma

$$dx = \left(\alpha - \frac{1}{2}\sigma^2\right)dt + \sigma dz$$

where dz is a simple Brownian motion. Then $\mu = \alpha - \frac{1}{2}\sigma^2$ and $b = \ln(y_{fl})$, so that (21) becomes

$$T(y_0) = \begin{cases} \frac{1}{\alpha - \frac{1}{2}\sigma^2}\ln\left(\frac{y_{fl}}{y_0}\right), \text{ if } \alpha \geq \frac{1}{2}\sigma^2, \\ \infty, \text{ if } \alpha < \frac{1}{2}\sigma^2 \end{cases}.$$

Replacing y_{fl} from (8) we find the expression (11) in the main text.