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Coordinating choice in partial cooperative equilibrium

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Abstract

In this paper we consider symmetric aggregative games and investigate partial cooperation between a portion of the players that sign a cooperative agreement and the rest of the players. Existence results of partial cooperative equilibria are obtained when the players who do not sign the agreement play a Nash equilibrium game having multiple solutions. Some applications in the supermodular case are discussed.

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1. Introduction

As in many fields of Economics, the formation of coalitions (i.e. cooperation) is an important topic. The classical approach is in the setting of cooperative games with the core and related solutions concepts (the stable set, the nucleolus, the bargaining set, the kernel). A more recent approach of coalition formation is in the setting of a mixed framework of both cooperative and non-cooperative games. Players within coalitions cooperate but coalitions act noncooperatively with each other. Some examples in Economics are cartels, R& D agreement, joint venture, environmental issues. See, for example, Yi (1997), Ray and Vohra (1999), Finus (2001), Montet and Serra (2003) and the references therein.

We deal with the concept of partial cooperative equilibrium. We suppose that a portion of the players (signatories) decide to cooperate and sign a cooperative agreement, the rest of the players (non-signatories) choose their strategies by playing a non-cooperative game, i.e. by solving a Nash equilibrium problem.

The definition of partial cooperative equilibrium has been presented in Mallozzi and Tijs (2008a) in the context of symmetric potential games, i.e. a subclass of symmetric strategic games with a potential in the sense of Monderer and Shapley (1996). An existence result of partial cooperative equilibria has been obtained for symmetric potential games by assuming the uniqueness of the non-signatories optimal strategy for any possible decision taken by the signatories and by using concavity-like assumptions. The uniqueness assumption has been removed in Mallozzi and Tijs (2008b) and a definition of partial cooperative equilibrium, where the Nash equilibrium problem of the non-signatories may have also different solutions, has been presented for non symmetric games depending on a selection choice in the set of equilibria.

In this paper we avoid the uniqueness assumption and, in order to reduce coordination problems, suppose that the non-signatories choose among the others only the symmetric Nash equilibria. We deal with aggregative games (Dubey et al. 1980; Corchon 1994), i.e. games having the payoffs depending only on individual strategies and an aggregate of all strategies. Several concrete situations correspond to these games, i.e. public good games (Batina and Ihori 2005), global emission games (Finus 2001).

We present the generalized definition of partial cooperative equilibrium, then specific examples are given and the existence question is investigated.

2. Coordinating Choice Model

Let $\Gamma =< n; X; f_1, ..., f_n >$ be an n-person normal form game with player set $I = \{1, 2, ..., n\}$, with the same strategy space X for each player $i \in I$ and where $f_i \colon X^n \mapsto \mathcal{R}$ is the payoff function of player $i \in I$. If player i chooses $x_i \in X$, then he obtains a profit $f_i(x_1, ..., x_n)$. Each player wants to maximize his own profit. We denote by x_{-i} the vector $(x_1, ..., x_{i-1}, x_{i+1}, ..., x_n) \in X^{n-1}$. We deal with aggregative games, i.e. games Γ for which there exist a function $f \colon X \times \mathcal{R} \mapsto \mathcal{R}$ and a function $g \colon X^n \mapsto \mathcal{R}$ (called aggregator) such that for all i and for all $x = (x_1, ..., x_n) \in X^n$ we have $f_i(x_1, \ldots, x_n) = f(x_i, g(x))$. We denote by $\Gamma =< n; X; f; g >$ an aggregative game. In the following we assume that $\Gamma =< n; X; f; g >$ is a symmetric aggregative game where:
a_1) X is a closed real interval;
a_2) f, g are continuous functions;
a_3) g is a symmetric function.

By assumption a_3) we have $g(x) = g(x')$ for all permutations x' of the vector $x \in X^n$, then $f_i(x_i, x_{-i}) = f_j(x'_j, x'_{-j})$ so Γ is a symmetric game. If noncooperative behavior is assumed between the n players, the equilibrium solution considered is the well known concept of Nash equilibrium. Let us denote by NE the set of the Nash equilibrium vectors of the game Γ, and by SNE the set of the symmetric Nash equilibrium vectors of the game Γ, i.e. vectors with identical components.

We suppose now that a portion of the n players may sign a cooperative agreement. Let $P_{k+1}, ..., P_n$ be the players acting in a cooperative way and $P_1, ..., P_k$ the players acting in a noncooperative way, for each $k = 0, ..., n$. We assume that the last $n - k$ players (cooperating players or signatories) use the same strategy, i.e. $x_{k+1} = x_{k+2} = ... = x_n = y$ for $y \in X$. This assumption is common in some concrete situations, for example in International Environmental Agreement it means the available level of a certain gas emission and countries sign the agreement by choosing the same strategy.

For simplicity we will denote $(y, ..., y) \in X^k$ by $y_{\mathbf{k}}$, for any $y \in X$ and $k = 0, ..., n$. The first k players (noncooperating players or non-signatories) with payoffs

$$f(x_i, g(x_1, ..., x_k, y_{\mathbf{n-k}}))$$

for any $i = 1, ..., k$ do not cooperate and choose a Nash equilibrium profile.

For each $k = 0, ..., n$ and for all $y \in X$, let us consider the normal form game $\Gamma_k(y) =< k; X; f(\cdot, y), g(\cdot, y) >$, i.e. the k-person game with the same strategy space X for each player and payoff function of player i, for $i = 1, ..., k$, given by $f(x_i, g(x_1, ..., x_k, y_{\mathbf{n-k}})$ for any $y \in X$. In order to define a concept of partial cooperative equilibrium for the game Γ, we denote by $NE_k(y)$ (resp. $SNE_k(y)$) the set of the (resp. symmetric) Nash equilibrium vectors of the game $\Gamma_k(y)$ for any $y \in X$.

Now, the uniqueness of the Nash equilibria of the non-signatories not always occurs and their multiplicity leads to coordination failures, as intensively studied by many authors (Cooper 1999). Here we suppose that the non-signatories select in the set of Nash equilibria only the symmetric ones in order to reduce coordination problems. Moreover, there may be still a coordination problem if the set $SNE_k(y)$ is not single-valued for a $y \in X$. We suppose that the signatories maximize their profit also in the set of the non-signatories optimal reaction, as specified in the following.

A vector $x(k) = (x^N_{\mathbf{k}}, \xi^C_{\mathbf{n-k}}) \in X^n$ such that

$$\xi^C \in \operatorname{argmax}_{y \in X} \left(\max_{x_{\mathbf{k}} \in SNE_k(y)} f(y, g(x_{\mathbf{k}}, y_{\mathbf{n-k}})) \right)$$

and $x^N_{\mathbf{k}}$ any symmetric equilibrium in $SNE_k(\xi^C)$, is called a *partial cooperative equilibrium* of the game Γ where $n - k$ players sign the agreement.

Let us note that in the context of hierarchical two-stage games, the above definition corresponds to the concept of strong hierarchical Nash equilibrium widely studied in the literature (for example Leitmann 1978; Luo et al. 1996 and the references therein). The definition of partial cooperative equilibrium has been given by Mallozzi and Tijs (2008a) for symmetric potential games by considering a unique symmetric Nash equilibrium for non-signatories, together with an existence result. Here we remove the uniqueness assumption in a larger class of games, namely symmetric aggregative games. In fact, if a symmetric aggregative game $\Gamma =< n; X; f; g >$ displays separability, i.e. for all i we have $f(x_i, g(x)) = B(x_i) - C(g(x))$ for some functions B and C, then the game turns out to be a symmetric potential game.

EXAMPLE 2.1 *Let us consider $n = 4$, $X = [0, 1]$ and the following payoffs*

$$f_i(x_1, x_2, x_3, x_4) = \Pi_{j=1}^4 x_j, \quad i = 1, 2, 3, 4$$

The game is a symmetric aggregative game with $f(x_i, g(x)) = g(x) = \Pi_{j=1}^4 x_j$.

If two of the four players cooperate, the rest of the players play a two-player noncooperative game with payoffs

$$f_1(x_1, x_2, y, y) = f_2(x_1, x_2, y, y) = x_1 x_2 y^2$$

for any $y \in X$, that admits two symmetric equilibria $SNE_2(y) = \{(0, 0), (1, 1)\}$ for $y \neq 0$ and infinite symmetric equilibria $SNE_2(0) = \{(x, x),\ x \in [0, 1]\}$

for $y = 0$. The signatory problem then is to maximize the function y^2 on $[0,1]$ and a partial cooperative equilibrium is $x(2) = (1,1,1,1)$.

Example 2.2 *Let us consider $n = 4$, $X = [0,1]$ and the following payoffs*

$$f_i(x) = x_i + x_i^2/2 - (x_1 + x_2 + x_3 + x_4)^2/2, \quad i = 1,2,3,4$$

The game is a symmetric aggregative game with $g(x) = (x_1+x_2+x_3+x_4)^2/2$ and $f(x_i, g(x)) = x_i + x_i^2/2 - g(x)$.

If two of the four players cooperate, the rest of the players choose a Nash equilibrium of the two-player game with payoffs

$$f_i(x_1, x_2, y, y) = x_i + x_i^2/2 - (x_1 + x_2 + 2y)^2/2, \; i = 1,2.$$

For $y > 1/2$ the game admits the only symmetric Nash equilibrium $(0,0)$; for $y = 1/2$ the set of equilibria is $NE_2(1/2) = \{(t,0),(0,t), t \in [0,1]\}$; for $y < 1/2$ the set $NE_2(y) = \{(1,0),(1-2y,1-2y),(0,1)\}$. In this case, for all $y \leq 1/2$, turns out that the symmetric Nash equilibrium is $SNE_2(y) = \{(1-2y,1-2y)\} \subset NE_k(y)$. The signatories problem will be to maximize the function $y + y^2/2 - (2-4y+2y)^2/2$ if $y \in [0,1/2]$, $y + y^2/2 - (2y)^2/2$ if $y \in]1/2,1]$ and $x(2) = (0,0,1/2,1/2)$ is the partial cooperative equilibrium.

In order to have an existence result, we need some properties of the correspondence NE_k mapping $y \in X$ into the set of the Nash equilibria $NE_k(y)$ and also of the correspondence SNE_k mapping $y \in X$ into the set of the symmetric Nash equilibria $SNE_k(y)$. First of all let us recall some definitions. Recall that a correspondence T from X to Y (X,Y topological spaces) is (sequentially) closed at $x \in X$ if, for any sequence (x_m) of X converging to $x \in X$ and any sequence (y_m) of Y converging to $y \in Y$ such that $y_m \in T(x_m)$ for all $m \in N$, we have $y \in T(x)$ (Aubin and Frankowska 1990). T is (sequentially) closed on X if it is closed at x, for every $x \in X$.

Proposition 2.1. *Let $\Gamma =< n; X; f, g >$ be a symmetric aggregative game s.t. $SNE_k(y) \neq \emptyset$, for any $y \in X$; under assumptions a_1), a_2) and a_3), the correspondences NE_k and SNE_k are closed on X.*

Proof. Let $y \in X$ and (y_m) a sequence in X such that $y_m \to y$, and let $(\eta_{1m}, ..., \eta_{km})$ a converging sequence of X^k, $(\eta_{1m}, ..., \eta_{km}) \to (\eta_1, ..., \eta_k)$ such that $(\eta_{1m}, ..., \eta_{km}) \in NE_k(y_m)$ for all $m \in \mathcal{N}$. We know that

$$f(\eta_{im}, g(\eta_{1m}, ..., \eta_{km}, (y_m)_{\mathbf{n-k}})) \geq f(x_i, g(\eta_{1m}, .., x_i, .., \eta_{km}, (y_m)_{\mathbf{n-k}}))$$

for any $x_i \in X$ and any $i = 1, ..., k$. By using assumption a_2), we have

$$f(\eta_i, g(\eta_1, ..., \eta_k, y_{\mathbf{n-k}})) \geq f(x_i, g(\eta_1, .., x_i, .., \eta_k, y_{\mathbf{n-k}}))$$

i.e. $(\eta_1, ..., \eta_k) \in NE_k(y)$.

Let us define the correspondence DIAG_k mapping to y the set $\{(x_1, ..., x_k) \in X^k,\ s.t.\ x_1 = x_2 = ... = x_k\}$. The correspondence SNE_k mapping to any $y \in X$ the set of symmetric Nash equilibria is the intersection correspondence between the correspondence NE_k and the correspondence DIAG_k, both closed on X. So that, SNE_k is closed on X (Border 1989). □

By assumptions $a_1)$, $a_2)$ and by using Berge's theorem, the marginal function $\max_{x_{\mathbf{k}} \in SNE_k(y)} f(y, g(x_{\mathbf{k}}, y_{\mathbf{n-k}}))$ is upper semicontinuous on X.

PROPOSITION 2.2 *Let $\Gamma =< n; X; f, g >$ be a symmetric aggregative game such that $SNE_k(y) \neq \emptyset$, for any $y \in X$; under assumptions $a_1)$, $a_2)$ and $a_3)$, there exists a partial cooperative equilibrium $x(k) = (x^N_{\mathbf{k}}, \xi^C_{\mathbf{n-k}}) \in X^n$.*

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