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A note on small-sample correction for hypothesis testing on cointegrating vectors: recursive Monte Carlo analysis

Takamitsu Kurita
Fukuoka University

Abstract

This note conducts recursive Monte Carlo experiments on the Bartlett correction for a likelihood-based test on cointegrating vectors. The experiments show that the correction can reduce size distortions even in situations where regularity conditions for I(1) cointegration analysis are satisfied only marginally.

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1 Introduction

This note conducts recursive Monte Carlo experiments on the Bartlett correction for a likelihood-based test on cointegrating vectors. The introductory section briefly reviews the literature on cointegrated vector autoregressive (VAR) models and then describes the most significant aspect of this note.

Economic time series data often exhibit non-stationary behaviour and thus need to be regarded as integrated processes rather than stationary. A cointegrated VAR model introduced by Johansen (1988, 1996) therefore plays a critical role in time series econometrics. The cointegrated VAR model usually assumes its innovation process to be Gaussian, so that a likelihood-based inference on cointegrating vectors is feasible. See Juselius (2006) and Kurita (2007), *inter alia*, for empirical macroeconomic research using cointegrated VAR models.

It is known that a likelihood-based test for a hypothesis on cointegrating vectors tends to suffer from size distortions when the number of observations is small. Various methods for small-sample correction are therefore introduced in the literature, such as F-type test adjustments (Podivinsky, 1992) and the use of bootstrap methods (Fachin, 2000). Johansen (2000), in particular, is a noteworthy achievement on the Bartlett correction — several correction factors are analytically derived from the expansion of log-likelihood ratio ($\log LR$) test statistics. Simulation studies by Johansen (2000), Canepa (2006), and Omtzigt and Fachin (2006) demonstrate that the Bartlett correction is useful for improving the small-sample performance of a likelihood-based test on cointegrating vectors.

Although the Bartlett correction is judged to be a useful tool, a detailed simulation analysis of the correction using a recursive Monte Carlo technique seems to be missing in the literature. Conducting Monte Carlo experiments in a recursive way allows us to evaluate how fast the required convergence is accomplished. See also Doornik (2005) for the validity of a recursive method in enhancing the generality of Monte Carlo experiments. Such recursive experiments, therefore, provide additional information on the usefulness of the correction. It is also of interest, in the context of applied research, to inspect how effective the Bartlett correction is in cases where regularity conditions for $I(1)$ cointegration analysis are only marginally fulfilled, such as weak adjustment and near $I(2)$ cases. This note thus carries out recursive Monte Carlo experiments on these marginal cases.

Recursive Monte Carlo experiments confirm that the Bartlett correction is very useful for reducing size distortions when the regularity conditions are all fulfilled *i.e.* the speed of convergence toward nominal significance levels is satisfactorily fast. When some of the regularity conditions are marginally satisfied, the convergence rate is affected and the performance of the Bartlett-corrected test turns worse than that when the conditions are fully satisfied. The overall performance is, however, still much better than the standard test without using the correction. These results can be treated as evidence indicating that the use of Bartlett correction is recommended regardless of whether the regularity conditions are fully or marginally satisfied. Evaluating the size-adjusted power of the tests, using various data generation mechanisms, is beyond the scope of this note, but should be investigated in future research to reinforce the argument of this note.

This note is organised as follows. Section 2 reviews a cointegrated VAR model and

Bartlett correction for a $\log LR$ test on cointegrating vectors. Section 3 then conducts recursive Monte Carlo experiments on the $\log LR$ tests with and without the Bartlett correction. Concluding remarks are provided in Section 4. All the numerical analyses and graphics in this paper use *Ox* (Doornik, 2006) and *OxMetrics / PcGive* (Doornik and Hendry, 2006). As a notational convention in this note, an orthogonal complement for a certain matrix a is defined as $a_\perp$ with full column rank and $a'_\perp a = 0$, so that a combined matrix $(a, a_\perp)$ is of full rank.

2 Cointegrated VAR Model and Bartlett Correction

This section reviews a cointegrated VAR model and the Bartlett correction for a hypothesis on the cointegrating vector. The main references are Johansen (1996, 2000). Consider an unrestricted VAR(k) model for a p-dimensional time series X_t conditional on the initial values $X_{-k+1}, ..., X_0$ as follows:

$$\Delta X_t = (\Pi, \Pi_c) \begin{pmatrix} X_{t-1} \\ 1 \end{pmatrix} + \sum_{i=1}^{k-1} \Gamma_i \Delta X_{t-i} + \varepsilon_t, \quad \text{for} \quad t = 1, ..., T, \tag{1}$$

where Π, $\Gamma_i \in \mathbf{R}^{p\times p}$ and $\Pi_c \in \mathbf{R}^p$ all vary freely. In order to find the Granger-Johansen representation of (1), which provides a basis for an $I(1)$ cointegrated VAR analysis, three regularity conditions need to be introduced. These are given in Assumption 2.1.

Assumption 2.1 *(cf. Theorem 4.2 in Johansen, 1996)*

1. *The characteristic roots obey the equation* $|A(z)| = 0$, *where*

$$A(z) = (1 - z) I_p - \Pi z - \sum_{i=1}^{k-1} \Gamma_j (1 - z) z^i,$$

and the roots satisfy $|z| > 1$ *or* $z = 1$.

2. $(\Pi, \Pi_c) = \alpha(\beta', \gamma')$ *for* $\alpha, \beta \in \mathbf{R}^{p\times r}$ *and* $\gamma' \in \mathbf{R}^r$ *with* $p > r > 0$.

3. *rank* $(\alpha'_\perp \Gamma \beta_\perp) = p - r$ *for* $\Gamma = I_p - \sum_{i=1}^{k-1} \Gamma_i$.

The first condition ensures that the VAR process is neither explosive nor seasonally cointegrated, and the second is a reduced rank condition, implying that there are at most r cointegrating relations. The third condition prevents the process from being $I(2)$ or of higher order. This note is interested in cases where either the second or third condition (or both) is satisfied only marginally. A set of vectors α are referred to as adjustment vectors, while β are called cointegrating vectors or cointegrating parameters. Let $\beta^{*\prime} = (\beta', \gamma')$ and $X^*_{t-1} = \left(X'_{t-1}, 1\right)'$ for future reference.

Exploring long-run economic relationships in the data is often of primary research interest in applied macroeconomics. Applied economists therefore attach importance to testing theory-consistent restrictions on cointegrating vectors. Restrictions on cointegrating vectors can be formulated in a number of ways, and the simplest formulation of them is H_0: $\beta = G\varphi$, where G is a $p \times s$ dimensional known matrix for $s \geq r$ and φ represents $s \times r$ dimensional unknown parameters to be estimated. Choosing the cointegrating rank r using Johansen's procedure, one can then construct the $\log LR$ test statistic for the null hypothesis of H_0 against the alternative of the cointegrating rank r. See Johansen (1996, Ch.7) for details of the testing procedure. The test statistic is denoted as $\log Q$ and referred to as *the standard test* in this note.

A corrected version of the test statistic derived by Johansen (2000) is $\log Q / \left(1 + T^{-1}B\right)$, where B denotes the Bartlett correction factor. The factor is explicitly given by

$$B = \frac{1}{2}(p + s - r + 3) + kp + \frac{1}{r}\left[(2p + s - 3r + 1)\, \upsilon(\alpha) + 2c(\alpha)\right],$$

where

$$\begin{aligned} \upsilon(\alpha) &= tr\left\{V_\alpha\right\}, \\ c(\alpha) &= tr\left\{P\left(I_{r+(k-1)p} + P\right)^{-1} V_\alpha\right\} \\ &\quad + tr\left\{\left[P \otimes \left(I_{r+(k-1)p} - P\right) V_\alpha\right]\left[I_{r+(k-1)p} \otimes I_{r+(k-1)p} - P \otimes P\right]^{-1}\right\}, \end{aligned}$$

for

$$\begin{aligned} V_\alpha &= \left(\alpha'\Omega^{-1}\alpha\right)^{-1} \Sigma_{\beta\beta}^{-1}, \\ P &= \begin{pmatrix} I_r + \beta'\alpha & \beta'\Gamma_1 & \dots & \beta'\Gamma_{k-2} & \beta'\Gamma_{k-1} \\ \alpha & \Gamma_1 & \dots & \Gamma_{k-2} & \Gamma_{k-1} \\ 0 & I_p & \dots & 0 & 0 \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \dots & I_p & 0 \end{pmatrix}, \end{aligned}$$

and $\Sigma_{\beta\beta}$ denotes variance of $\beta^{*\prime} X_t^*$ conditional on $\Delta X_t, ..., \Delta X_{t-k+2}$. The corrected test statistic, $\log Q / \left(1 + T^{-1}B\right)$, is referred to as *the Bartlett-corrected test* in this note.

3 Recursive Monte Carlo Experiments

This section conducts recursive Monte Carlo experiments on both of the standard and Bartlett-corrected tests. The data generation process (DGP) is specified as follows:

$$\begin{aligned} \Delta X_{1,t} &= a\left(X_{1,t-1} - X_{2,t-1} + 0.1\right) + 0.1\Delta X_{1,t-1} + \varepsilon_{1,t}, \\ \Delta X_{2,t} &= b\Delta X_{2,t-1} + \varepsilon_{2,t}, \\ \begin{pmatrix} \varepsilon_{1,t} \\ \varepsilon_{2,t} \end{pmatrix} &\sim IN\left[0, \begin{pmatrix} 0.1 & 0.05 \\ 0.05 & 0.1 \end{pmatrix}\right]. \end{aligned}$$

The number of observations, denoted by T, increases one by one from 40 to 100, so that the experiments correspond to typical empirical macroeconomic research using quarterly data; the initial point $T = 40$ coincides with the number of quarterly data available for 10 years, while the end point $T = 100$ agrees with that for 25 years. The number of Monte Carlo replications is $10,000$, and the null hypothesis H_0 needs to reflect the DGP

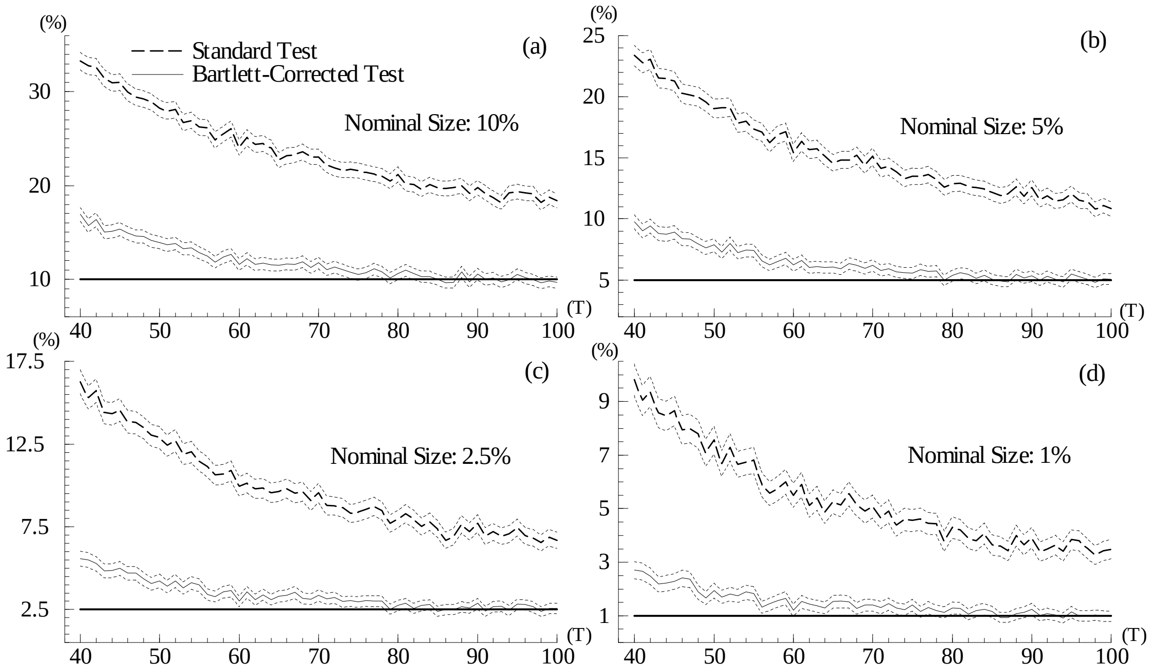

Figure 1: Benchmark Case

so $G = (1, -1)'$. The parameters in the DGP, a and b, vary according to the following four cases:

1. Benchmark case: $a = -0.2$ and $b = 0.2$
2. Weak adjustment case: $a = -0.05$ and $b = 0.2$
3. Near $I(2)$ case: $a = -0.2$ and $b = 0.9$
4. Weak adjustment and near $I(2)$ case: $a = -0.05$ and $b = 0.9$

The benchmark case clearly fulfills all the regularity conditions in Assumption 2.1, while the weak adjustment case marginally satisfies the second condition and the near $I(2)$ case marginally fulfills the third condition. The final case corresponds to a combination of the weak adjustment and near $I(2)$ cases.

Recursive empirical sizes (rejection frequencies) of both standard and Bartlett-corrected tests are presented in figures of this note; the empirical sizes of the standard test are expressed as thick dotted lines, while those of the Bartlett-corrected test are represented by solid lines. The figures also provide 95% confidence bands, denoted by thin dotted lines. The objective of calculating empirical sizes recursively is to see how size distortions vary according to T. Four nominal sizes are under investigation: (a)10%, (b)5%, (c)2.5% and (d)1%.

Figure 1 shows various recursive empirical sizes for the benchmark case. According to the figure, all of the empirical sizes of the Bartlett-corrected test converge to the corresponding nominal sizes as the number of observations increases; they coincide with the corresponding nominal sizes around $T = 80$, a typical number of observations available in applied macroeconomic research. In contrast, the empirical sizes of the standard test converge much more slowly to the corresponding nominal levels; the standard test suffers

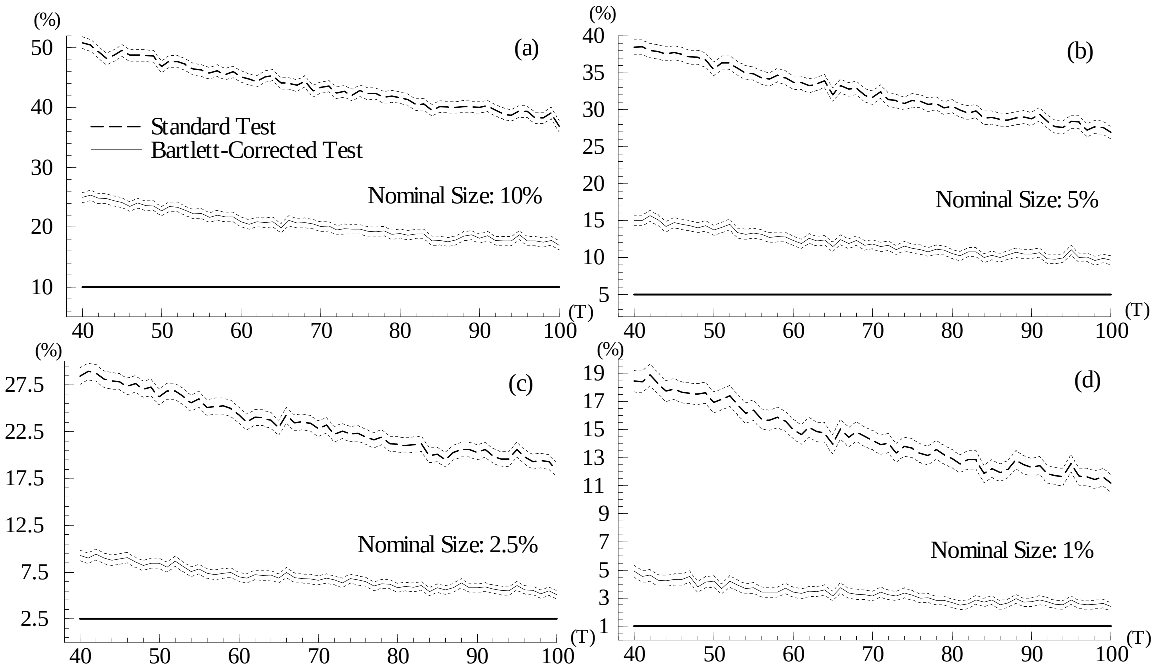

Figure 2: Weak Adjustment Case

from serious size distortions even around $T = 80$. Figure 1 demonstrates the validity of the Bartlett correction when the regularity conditions are fully satisfied.

Figure 2 displays recursive empirical sizes for the weak adjustment case. Size distortions in Figure 2 are uniformly larger than those in Figure 1 — evidence that the magnitude of the adjustment parameter can have a significant effect on the size properties of both of the standard and Bartlett-corrected tests. However, size distortions of the Bartlett-corrected test are much smaller than those of the standard test, suggesting that the correction is still useful even in such a case as the adjustment mechanism is not strong.

Figure 3 then shows recursive empirical sizes for the near $I(2)$ case. In contrast to Figure 2, the empirical sizes of the Bartlett-corrected test tend to lie below the corresponding nominal levels *i.e.* the Bartlett-corrected test in this case is liable to be conservative. It is worth bearing in mind that the Bartlett correction can give rise to over-correction like in this case. However, size control is a fundamental requirement in a classical statistical inference and a conservative test is therefore favourable in comparison with a size-distorted test. According to Figure 3, the standard test again suffers from size distortions, thus lending weight to the validity of the Bartlett-corrected test in this case as well.

Finally, Figure 4 displays recursive empirical sizes for the case where weak adjustment and near $I(2)$ roots are both involved in the DGP. It seems that impacts of weak adjustment and near $I(2)$ roots cancel out each other, leading to an improvement in the overall performance of both of the tests. It is again found that size distortions of the Bartlett-corrected test are uniformly smaller than those of the standard test, in line with the preceding figures.

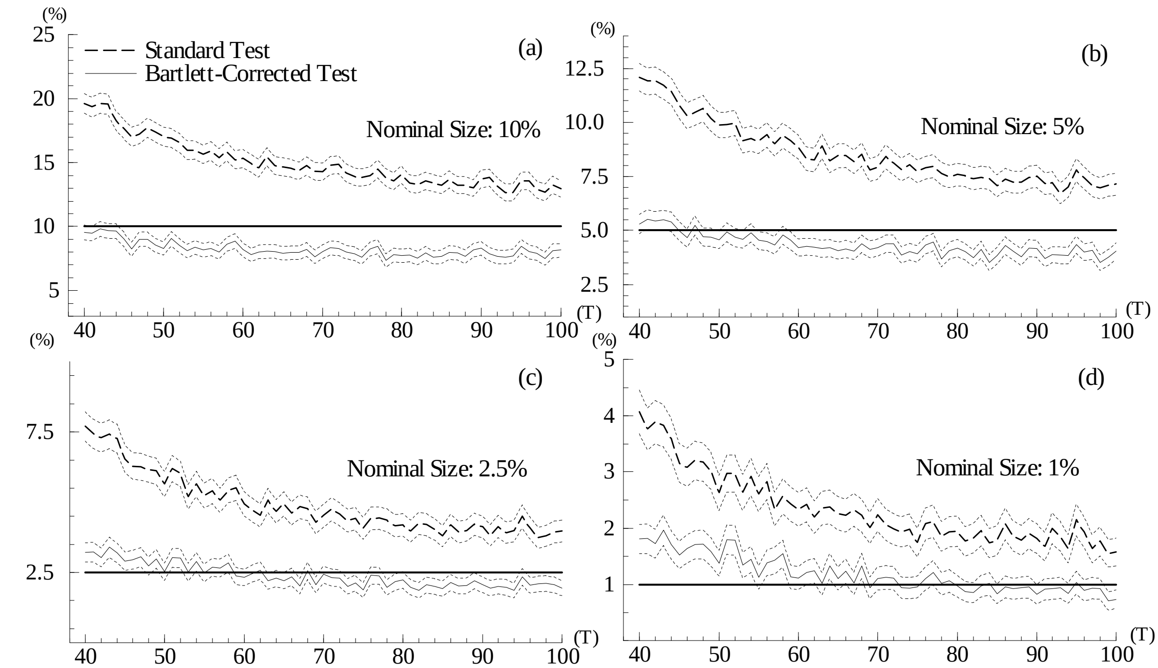

Figure 3: Near $I(2)$ Case

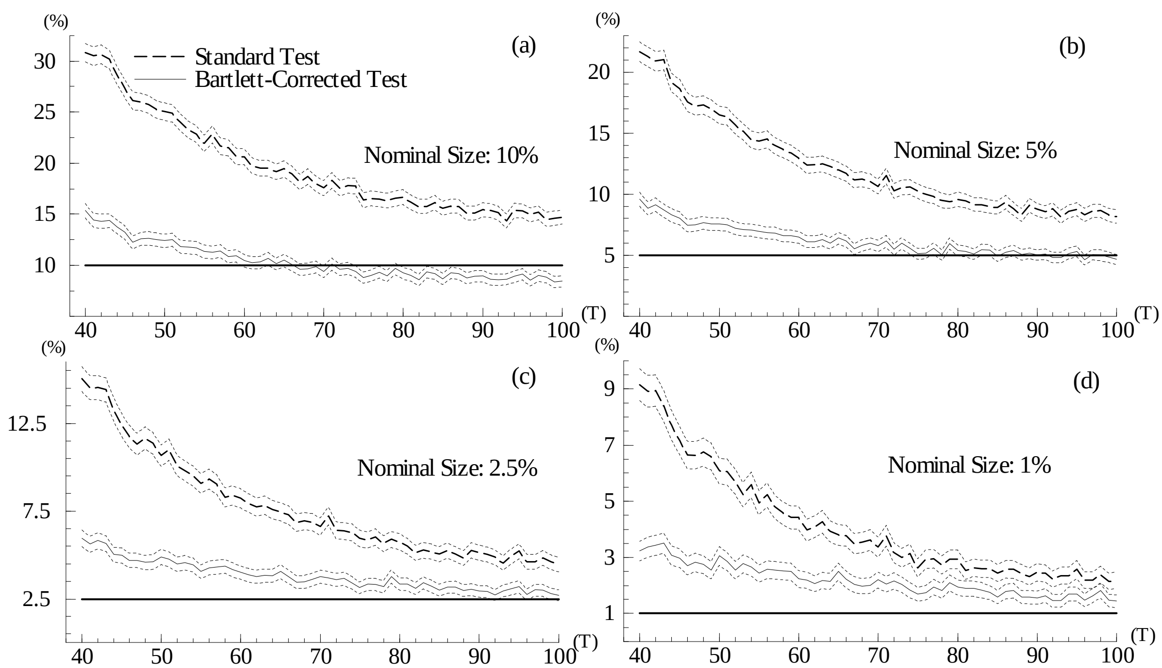

Figure 4: Weak Adjustment and Near $I(2)$ Case

4 Concluding Remarks

This note carries out recursive Monte Carlo experiments on the Bartlett correction for a likelihood-based test on cointegrating vectors. The experiments demonstrate that the correction can be useful for reducing size distortions even in circumstances where regularity conditions for $I(1)$ cointegration analysis are satisfied only marginally.

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