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Equilibrium in Matching Models with Employment Dependent Productivity

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Abstract

In a standard search and matching framework, the labor market presents frictions while in the competitive product market the demand is infinitely elastic.To have a more realistic framework, some models abandon the assumption of infinite elasticity and consider a two-tier productive scheme in the goods market. In this paper, I establish the conditions that are sufficient for the existence and the uniqueness of a steady-state equilibrium for this kind of models. I also notice that some standard assumptions about the production and matching technology (a Cobb-Douglas function) do not fulfill such conditions and so may hinder the existence of an equilibrium.

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1 Introduction

In a standard search and matching model, the labor market presents frictions, whereas in the perfectly competitive product market the demand is infinitely elastic, so that an increase in supply does not affect the equilibrium price (Mortensen and Pissarides 1999, and Pissarides 2000).

In order to introduce a more realistic framework for the goods market, some scholars (e.g. Joseph, Pierrard, and Sneessens 2004, and Pierrard 2005, and Cahuc and Zylberberg 2004, p. 618-622) abandon the assumption of an infinite elasticity of demand and consider a two-tier productive scheme. Different types of workers (usually, low-skilled and high-skilled ones) are hired in intermediate good sectors. Such goods face a decreasing demand from a final representative firm that produces the unique consumption good[1]. Papers of this kind make often use of numerical simulations and scant attention is paid to analytical properties.

This paper takes a different stand. I consider a simplified framework in which there are only two intermediate sectors and I look for the conditions under which a (unique) steady-state equilibrium exists. Uniqueness is guaranteed by the assumption of constant returns to scale (henceforth, CRS) in the final good function, whereas a sufficient condition for the existence concerns the difference between the marginal productivity and the income received when unemployed. If it is positive as the levels of labor market tightness in both sectors tend to zero, then an equilibrium exists.

Since this condition is not fulfilled by imposing a Cobb-Douglas technology both in the matching and in the production function, introducing such a functional form may hinder the existence of an equilibrium.

2 The Model

2.1 Production Technology

Assume an economy with one final good (the numeraire), two intermediate goods sectors and two types m and n of infinitely-lived and risk-neutral workers. The goods markets are perfectly competitive. Each producer of an intermediate good hires only one type of worker. Moreover, every m-skilled (respectively, n-skilled) employee produces one unit of the intermediate good m (resp. n). So E_i ($i \in \{m, n\}$) denotes both the amount of the i intermediate good produced and the number of employees in the i-th sector. The final good production function exhibits CRS and is written as:

$$Y = F(E_m, E_n), \;\; \text{with } \frac{\partial F}{\partial E_i} > 0 \text{ and } \frac{\partial^2 F}{\partial {E_i}^2} < 0, \quad i \in \{m, n\}. \tag{1}$$

The two inputs are p-substitutes ($\frac{\partial^2 F}{\partial E_m \partial E_n} > 0$).[2] Let p_i denote the real price of the intermediate good i. Cost minimization in the final sector leads to $p_i = \partial F(E_m, E_n)/\partial E_i$, with $i \in \{m, n\}$. Further, the value of home production is denoted by $b_i > 0$.

[1] Acemoglu (2001) has also constructed a similar model, but with one decisive difference: workers are identical ex ante and can be employed in high-paid or low-paid jobs.

[2] I also assume one Inada condition: $\lim_{E_i \to 0} \frac{\partial F}{\partial E_i} = +\infty$. Imposing a condition as $E_i \to +\infty$ is useless, since in this model the upper bounds of both inputs are given by the labor force that has a positive finite value.

2.2 Search Technology

The model is developed in steady state. Time is continuous and r denotes the discount factor. Each type of worker can be either unemployed or be employed in his sector. The labor market is perfectly segmented, meaning that every i-type worker can be hired only by firms in the i sector. The matching function is written respectively $M_i = m(U_i, V_i)$, with U_i being the number of unemployed people and V_i the number of job vacancies in sector i. It is assumed to be increasing, concave and homogeneous of degree 1. Labor market tightness is defined as $\theta_i \equiv \frac{V_i}{U_i}$. The job filling rate is $q(\theta_i) \equiv M_i/V_i = m(\frac{1}{\theta_i}, 1)$, $q'(\theta_i) < 0$, whereas the job finding rate is equal to $\alpha(\theta_i) \equiv M_i/U_i = \theta_i q(\theta_i)$, with $\alpha'(\theta_i) > 0$.[3] At an exogenous rate ϕ_i a match is destroyed. In steady state, the stocks of individuals in each position are constant. With an exogenous size of the labor force, L_i, the employment level is given by:

$$E_i = \mathbb{E}_i(\theta_i) \equiv \frac{\alpha(\theta_i)}{\phi_i + \alpha(\theta_i)} \cdot L_i, \quad i \in \{m, n\}. \tag{2}$$

Notice that $\mathbb{E}_i'(\theta_i) > 0$ and, from the condition in footnote 3, $\lim_{\theta_i \to +\infty} \mathbb{E}_i(\theta_i) = L_i$.

2.3 Vacancy Supply and Wage-Setting Curves

Once a worker finds a firm with a vacant job, a surplus of the match arises. The Nash bargaining solution is assumed in order to split the surplus, whereas a zero profit condition is imposed on the demand side of the market.

The characterization of the model is standard. Hence, I directly present the equilibrium equations (obtained by merging the so-called vacancy supply and wage-setting curves) and refer to Pissarides (2000, chapter 1) for the primitive conditions under which they are derived:

$$\mathbb{G}_i(\theta_i, \theta_j) \equiv (1-\beta_i)\left[\frac{\partial F}{\partial E_i}(\mathbb{E}_i(\theta_i), \mathbb{E}_j(\theta_j)) - b_i\right] - k_i\left(\frac{r+\phi_i}{q(\theta_i)} + \beta_i\theta_i\right) = 0, \tag{3}$$

with $i \in \{m, n\}$, $i \neq j$, k_i is the flow cost of posting a vacancy in units of final good, and β_i represents workers' bargaining power. Function $\mathbb{G}_i = 0$ is the equilibrium condition in labor market i and depends on θ_j only through the marginal productivity $\frac{\partial F}{\partial E_i}$. Differentiating $\mathbb{G}_i$ with respect to θ_i , I obtain:

$$\frac{d\mathbb{G}_i}{d\theta_i} = A_i + B_i, \tag{4}$$

with

$$A_i \equiv k_i\left[(r+\phi_i)\frac{q'(\theta_i)}{q(\theta_i)^2} - \beta_i\right] < 0 \quad \text{and} \quad B_i \equiv (1-\beta_i)\frac{\partial^2 F}{\partial {E_i}^2} \cdot \mathbb{E}_i'(\theta_i) < 0,$$

$i \in \{m, n\}$. I also differentiate $\mathbb{G}_i$ with respect to E_j:

$$\frac{d\mathbb{G}_i}{d\theta_j} \equiv C_{i,j} = (1-\beta_i)\frac{\partial^2 F}{\partial E_i \partial E_j} \cdot \mathbb{E}_j'(\theta_j) > 0 \qquad \text{with } i, j \in \{m, n\}, \, i \neq j. \tag{5}$$

[3]Moreover, I impose that $\lim_{\theta_i \to 0} q(\theta_i) = +\infty$ and $\lim_{\theta_i \to +\infty} \alpha(\theta_i) = +\infty$.

3 Equilibrium

3.1 Conditional Equilibrium in each market

The first existence result consists in showing under which assumptions there exists a θ_i such that $\mathbb{G}_i(\theta_i, \theta_j) = 0$ holds, *conditional on* θ_j.

Note that $\frac{d\mathbb{G}_i}{d\theta_i} < 0$ and $\lim_{\theta_i \to +\infty} \mathbb{G}_i(\theta_i, \theta_j) \to -\infty$ for the conditions imposed in footnote 3 and the limit behavior of equation (2). This result holds for any value of $\theta_j \in [0, +\infty)$. If $\lim_{\theta_i \to 0} \mathbb{G}_i(\theta_i, \theta_j) > 0 \; \forall \theta_j \in [0, +\infty)$, I could use the intermediate value theorem to prove the existence of a conditional equilibrium. However, I can compute such a limit only for values of θ_j that are not close to zero[4]. In fact, as both θ_i and θ_j (and consequently E_i and E_j) tend to 0, the input of the marginal productivity takes an indeterminate form[5] $\frac{0}{0}$ and it is not possible to ascertain the sign of this expression[6].

To rule out this possibility, two alternative assumptions are needed. Lemma 1 summarizes the results.

Lemma 1 *There always exists a* $\theta_i \in (0, +\infty)$ *that solves* $\mathbb{G}_i(\theta_i, \theta_j) = 0 \; \forall \theta_j \in [0, +\infty)$, $i, j \in \{n, m\}$, $i \neq j$ *if, alternatively,:*

1. $\lim_{\theta_i \to 0} E_i = \mathbb{E}_i(\theta_i) = 0$ *and* $\frac{\partial F}{\partial E_i}(\mathbb{E}_i, \mathbb{E}_j) > b_i \quad \forall \theta_i$ *and* $\theta_j \in [0, +\infty)$ *(except for* $\theta_i = \theta_j = 0$*) with* $i, j \in \{m, n\}$, $i \neq j$ *.*
2. $\lim_{\theta_i \to 0} E_i = \mathbb{E}_i(\theta_i) = \epsilon_i > 0$, *and* $\frac{\partial F}{\partial E_i}(\mathbb{E}_i = \epsilon_i, \mathbb{E}_j = \epsilon_j) > b_i$ *with* $i, j \in \{m, n\}$, $i \neq j$*. .*

Proof.

CASE 1

Since $\lim_{\theta_i \to 0} E_i = \mathbb{E}_i(\theta_i) = 0$, $i \in \{m, n\}$, the sets $\theta_i \in [0, +\infty)$ and $E_i \in [0, L_i]$ are respectively the domain and the range of the function $\mathbb{E}_i(\theta_i)$.
Consider the term inside the square brackets in (3) as $\theta_j = 0$; it is positive, decreasing in $\theta_i \in (0, +\infty)$, and tends to $\frac{\partial F}{\partial E_i}(L_i, \mathbb{E}_j = 0) - b_i > 0$ as $\theta_i \to +\infty$. The second term in (3) is a ray starting from the origin and that tends to $+\infty$ as $\theta_i \to +\infty$.
So, there exists a $\theta_i \in [0, +\infty)$ such that $\mathbb{G}_i(\theta_i, \theta_j = 0) = 0$. The same reasoning can be applied for any $\theta_j > 0$.

CASE 2

Since $\lim_{\theta_i \to 0} E_i = \mathbb{E}_i(\theta_i) = \epsilon_i > 0$, the domain and range of the function $\mathbb{E}_i(\theta_i)$ become respectively $\theta_i \in [0, +\infty)$ and $E_i \in [\epsilon_i, L_i]$, $i \in \{m, n\}$. Imposing $\frac{\partial F}{\partial E_i}(\mathbb{E}_i = \epsilon_i, \mathbb{E}_j = \epsilon_j) > b_i$ implies that $\lim_{\theta_i \to 0} \mathbb{G}_i(\theta_i, \theta_j) > 0 \; \forall \theta_j \geq 0$.
I can apply the intermediate value theorem and conclude that there exists a $\theta_i \in [0, +\infty)$ such that $\mathbb{G}_i = 0 \; \forall \theta_j \in [0, +\infty)$.

∎

[4]In this case, it tends to infinity for the Inada condition in footnote (2) and if $\lim_{\theta_i \to 0} E_i = \mathbb{E}_i(\theta_i) = 0$.

[5]Recall that the first derivatives of CRS functions can be expressed in terms of the ratio of the two inputs.

[6]On the contrary, there are no difficulties as $\theta_j \to +\infty$. This is because $\lim_{\theta_j \to +\infty} E_j = L_j$ and $\frac{\partial F}{\partial E_i}$ takes a positive finite value as $E_j = L_j$. So there exists a $\theta_i > 0$ that solves $\mathbb{G}_i(\theta_i, \theta_j \to +\infty) = 0$.

Some Examples
CASE 1: CES production function with $s > 1$.
Consider a CES production function with $s > 1$:

$$Y = \left[E_m^{\frac{s-1}{s}} + E_n^{\frac{s-1}{s}} \right]^{\frac{s}{s-1}} \quad \text{where} \quad \frac{\partial F}{\partial E_i}(E_i, E_j = 0) = 1 \quad i, j \in \{n, m\}, i \neq j.$$

For Lemma 1 (CASE 1), imposing $b_i < 1$ is sufficient to ensure the existence of a conditional equilibrium.
CASE 2: CES matching function with $s > 1$.
Consider a CES matching function:

$$M_i = \left[V_i^{\frac{s-1}{s}} + U_i^{\frac{s-1}{s}} \right]^{\frac{s}{s-1}} \quad \text{and} \quad \alpha(\theta_i) = \left[\theta_i^{\frac{s-1}{s}} + 1 \right]^{\frac{s}{s-1}} \quad i \in \{n, m\}.$$

If $s > 1$, $\lim_{\theta_i \to 0} E_i = \mathbb{E}_i(\theta_i) = \frac{L_i}{1+\phi_i}$. Imposing $\frac{\partial F}{\partial E_i}\left(\frac{L_i}{1+\phi_i}, \frac{L_j}{1+\phi_j}\right) > b_i$, $i, j \in \{n, m\}, i \neq j$, is sufficient to ensure the existence of a conditional equilibrium.
Cobb-Douglas technology.
A Cobb-Douglas production function $Y = aE_n^{\gamma} E_m^{1-\gamma}$ does not belong to CASE 1. In fact, $\frac{\partial F}{\partial E_i} = \gamma Y / E_i = 0$ as $E_j = 0$, $i, j \in \{n, m\}, i \neq j$.

Similarly, with a Cobb-Douglas matching function $M_i = aV_i^{\gamma} U_i^{1-\gamma}$, $\alpha(\theta_i) = a\theta_i^{\gamma}$, and $\lim_{\theta_i \to 0} E_i = \mathbb{E}_i(\theta_i) = 0$, so conditions of CASE 2 are not fulfilled.

Imposing a Cobb-Douglas formulation both in the the production and in the matching technology implies that the conditional equilibrium in market i may not be satisfied for values of θ_j close to 0. As we will see in the next paragraph, this may cause the absence of the general equilibrium.

3.2 General Equilibrium

I first apply the implicit function theorem. Using (4) and (5), I get:

$$\left.\frac{d\theta_i}{d\theta_j}\right|_{\mathbb{G}_i = 0} = -\frac{C_{i,j}}{A_i + B_i} > 0 \qquad \text{with } i \in \{m, n\}, i \neq j. \tag{6}$$

$\mathbb{G}_i = 0$ defines a monotonously increasing relationship in (θ_i, θ_j) space. At the general equilibrium, the values of labor market tightness solve the following system:

$$\begin{cases} G_n(\theta_n, \theta_m) = 0 \\ G_m(\theta_m, \theta_n) = 0 \end{cases} \tag{7}$$

Proposition 1 presents the results.

Proposition 1 *If the conditions in Lemma 1 are satisfied, a steady state equilibrium exists and is unique.*

Proof.
I denote $\theta_m = g_m(\theta_n)$ the explicit function of $\mathbb{G}_m(\theta_m, \theta_n) = 0$, and $\theta_m = g_n(\theta_n)$ the explicit function of $\mathbb{G}_n(\theta_n, \theta_m) = 0$. Both functions are monotonically increasing. Let also $\theta_n = g_m^{-1}(\theta_m)$ and $\theta_n = g_n^{-1}(\theta_m)$ denote their inverse functions.

EXISTENCE
For the conditions imposed in Lemma 1, there always exists a $\theta_m \in (0, +\infty)$ that solves $\mathbb{G}_m(\theta_m, \theta_n = 0) = 0$. This is tantamount to writing that $\theta_m = g_m(\theta_n)$ has a positive intercept in the vertical axis, that I denote with χ_m. Similarly, there always exists a $\theta_n \in (0, +\infty)$ that solves $\mathbb{G}_n(\theta_m = 0, \theta_n) = 0$, implying that $\theta_m = g_n(\theta_n)$ has a positive intercept in the horizontal axis, that I denote with χ_n.
Moreover, as noticed in footnote 6, there exists a $\theta_m \in (0, +\infty)$ that solves $\mathbb{G}_m(\theta_m, \theta_n \rightarrow +\infty) = 0$. So, $\lim_{\theta_n \rightarrow +\infty} g_m(\theta_n) = \Psi_m \in \Re^+$. Since there also exists a $\theta_n \in (0, +\infty)$ that solves $\mathbb{G}_n(\theta_m \rightarrow +\infty, \theta_n) = 0$, one gets that $\lim_{\theta_m \rightarrow +\infty} g_n^{-1}(\theta_m) = \Psi_n \in \Re^+$.
The domain and the range of $\theta_m = g_m(\theta_n)$ are respectively $[0, +\infty)$ and $[\chi_m, \Psi_m]$. The domain and the range of $\theta_m = g_n(\theta_n)$ are respectively $[\chi_n, \Psi_n]$ and $[0, +\infty)$. Since both functions are monotonously increasing, they must intersect at least once (see Figure 1).
UNIQUENESS
I define $H(\theta_n) \equiv g_m(\theta_n) - g_n(\theta_n)$. If $H(\theta_n)$ is a monotonic function in the neighborhood of the equilibrium steady-state, the equilibrium is unique.
Let θ_m^* and θ_m^* denote the equilibrium levels of tightness, $H'(\theta_n^*) = g_m'(\theta_n^*) - g_n'(\theta_n^*) < 0$ is a sufficient condition for the uniqueness of the equilibrium. This implies:

$$\frac{d\theta_m}{d\theta_n}\bigg|_{\mathbb{G}_n(\theta_n^*, \theta_m^*) = 0} > \frac{d\theta_m}{d\theta_n}\bigg|_{\mathbb{G}_m(\theta_m^*, \theta_n^*) = 0}. \tag{8}$$

From (6), one derives:

$$\frac{d\theta_m}{d\theta_n}\bigg|_{\mathbb{G}_n(\theta_n^*, \theta_m^*) = 0} = -\frac{B_n^* + A_n^*}{C_{n,m}^*} \tag{9}$$

$$\frac{d\theta_m}{d\theta_n}\bigg|_{\mathbb{G}_m(\theta_m^*, \theta_n^*) = 0} = -\frac{C_{m,n}^*}{B_m^* + A_m^*} \tag{10}$$

I multiply the numerator of (9) by the denominator of (10) and the numerator of (10) with the denominator of (9). I get four positive terms on the LHS and only one positive term on the RHS. For (8) to hold, the four positive terms on the LHS must be greater than the term on the RHS. One of the term on the LHS is:

$$B_m^* B_n^* = (1-\beta_m)(1-\beta_n)\frac{\partial^2 F}{\partial {E_m}^2}(\mathbb{E}(\theta_n^*), \mathbb{E}(\theta_m^*)) \cdot \frac{\partial^2 F}{\partial {E_n}^2}(\mathbb{E}(\theta_m^*), \mathbb{E}(\theta_n^*)) \cdot \mathbb{E}'(\theta_m^*)\mathbb{E}'(\theta_m^*) \tag{11}$$

The positive term on the RHS is:

$$C_{n,m}^* C_{m,n}^* = (1-\beta_m)(1-\beta_n)\left[\frac{\partial^2 F}{\partial E_m \partial E_n}(\mathbb{E}(\theta_m^*), \mathbb{E}(\theta_m^*))\right]^2 \cdot \mathbb{E}'(\theta_m^*)\mathbb{E}'(\theta_n^*). \tag{12}$$

Expressions (11) and (12) are equal because of the Euler's formula for functions with constant returns to scale, that is $\frac{\partial^2 F}{\partial {E_n}^2}\frac{\partial^2 F}{\partial {E_m}^2} = \left(\frac{\partial^2 F}{\partial E_n \partial E_m}\right)^2$. Then, inequality (8) is verified. The equilibrium is unique.

■

4 Final Remarks

A Cobb-Douglas technology both in the matching and in the production function may hinder the existence of a steady-state equilibrium. The reason is that the functions

$\theta_m = g_m(\theta_n)$ and $\theta_m = g_n^{-1}(\theta_n)$ may not exist for values of respectively θ_n and θ_m close to 0, point 1 of the existence proof cannot be established, and the two curves may not cross each other. Figure 2 illustrates this case.

This is not a rare possibility. Consider a standard parametrization such as the one performed by Cahuc and Zylberberg (2004, page 623) for this kind of models. The unit of time corresponds to one year, $r = 0.05$, $\beta_i = 0.5$, $\phi_i = 0.15$, $h_i = 0.1$ $\forall i$. The matching function is Cobb-Douglas with a parameter $\gamma_i = 0.5$ $\forall i$. Contrarily to Cahuc and Zylberberg (that consider a CES production function with elasticity of substitution equal to 1.5), I impose a Cobb-Douglas formulation even in the production technology. A_n is an efficiency parameter for workers in market n and it is equal to 1.5. For values of $b_m \geq 0.9$ and $b_n \geq 0.95$, no equilibrium exists[7].

Keeping the Cobb-Douglas formulation and imposing either $b_i = \rho_i w_i$ with $0 < \rho_i < 1$ or $b_i = 0$ $i \in \{n, m\}$ would not prevent from the absence of an equilibrium. The reason is that, even under these hypotheses, the solution of the equation $\mathbb{G}(\theta_i, \theta_j = 0) = 0$ is indefinite the existence proof cannot be made[8].

The results of Proposition 1 should be seen as a warning about the use of a standard Cobb-Douglas technology in models of this kind.

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[7] The file containing the numerical exercise is available on request.

[8] In other terms, the two functions may exist for values of $\theta_j \to 0$ (since as $b_i = 0$ or $b_i = \rho_i w_i$, it is sufficient that the marginal productivity is slightly positive for $\lim_{\theta_i \to 0} \mathbb{G}_i(\theta_i, \theta_j)$ to be positive) but are still not defined along the axis (i.e. as $\theta_j = 0$). One could not rule out the case that the only possible point of intersection of the two curves is the origin, that however does not belong to the domain of $g_m(\theta_n)$ and $g_n^{-1}(\theta_m)$. See Figure 2 on the right.

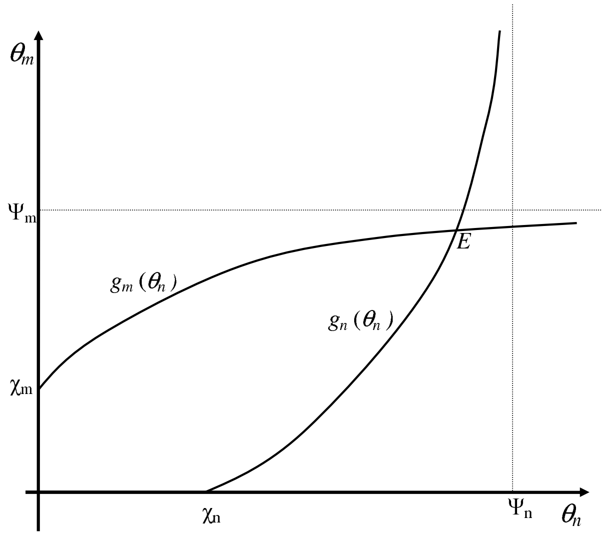

Figure 1: Existence and uniqueness of the equilibrium.

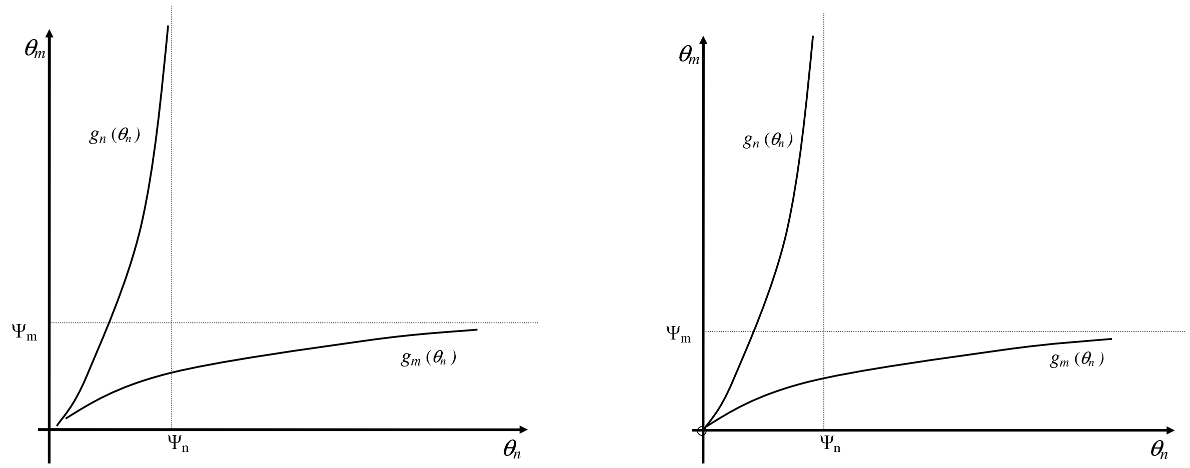

Figure 2: No equilibrium. On the Left: As $\theta_n \to 0$ (resp. $\theta_m \to 0$), the function $g_m(\theta_n)$ (resp. $g_n(\theta_n)$) is not defined. On the Right: At the origin, the functions $g_m(\theta_n)$ and $g_n(\theta_n)$ are not defined.