Economics Bulletin

Volume 29, Issue 3

Stochastic homothetically revealed preference for tight stochastic demand functions

Jan Heufer
TU Dortmund University and Ruhr Graduate School in Economics

Abstract

This paper strengthens the framework of stochastic revealed preferences introduced by Bandyopadhyay et al. (1999, 2004) for stochastic homothetically revealed preferences for tight stochastic demand functions.

I am grateful to my advisor Wolfgang Leininger for his support and comments and to an anonymous referee for helpful suggestions.
Citation: Jan Heufer, (2009) "Stochastic homothetically revealed preference for tight stochastic demand functions", *Economics Bulletin*, Vol. 29 no.3 pp. 2472-2477.
Submitted: Aug 18 2009. **Published:** September 30, 2009.

1 Introduction

Bandyopadhyay, Dasgupta, and Pattanaik (1999) initiated a line of investigation in which they explored choice behavior of a consumer who chooses in a stochastic fashion from different budget sets. In Bandyopadhyay, Dasgupta, and Pattanaik (2002) this approach was extended by an interpretation of tuples of deterministic demand functions of different consumers as a stochastic demand function. They define a weak axiom of stochastic revealed preference which is implied by but does not imply rationalizability in terms of stochastic orderings. In Bandyopadhyay, Dasgupta, and Pattanaik (2004), the authors show that if a consumer spends his entire wealth their weak axiom is equivalent to a restriction they call stochastic substitutability. Heufer (2008) transformed the problem of finding a probability measure on orderings over the available commodity bundles into a problem of finding a probability measure on orderings over the budgets on which choices are observed.

In this paper we strengthen the framework for rationalizability in terms of stochastic homothetic orderings. We focus on the case in which a consumer spends his entire wealth. It is shown that rationalizability in terms of stochastic homothetic orderings implies but is not implied by a homothetic version of stochastic substitutability.

2 Notation and Definitions

Let ℓ denote the number of commodities, $\ell \geq 2$, and let $\mathbb{R}_+$ and $\mathbb{R}_{++}$ denote the set of non-negative real numbers and positive real numbers, respectively. The consumption set is $\mathbb{R}^\ell_+$. A price-wealth situation is a pair $(p, w) \in Z$, where $Z = \mathbb{R}^\ell_{++} \times \mathbb{R}_{++}$ is the set of all possible price-wealth situations. Given a price-wealth situation (p, w), the budget set of a consumer is defined as $\left\{x \in \mathbb{R}^\ell_+ : w \geq p \cdot x\right\}$, where $\cdot$ denotes the dot product. The boundary of a budget is defined as $\left\{x \in \mathbb{R}^\ell_+ : w = p \cdot x\right\}$. The budget sets corresponding to price-wealth situations (p, w), (p', w'), etc. will be denoted $B(p, w)$, $B(p', w')$, etc.

Definition 1 (Bandyopadhyay et al. 2004) A *deterministic demand function* (DDF) is a rule d, which, for every $(p, w) \in Z$, specifies exactly one bundle x in $B(p, w)$. A *stochastic demand function* (SDF) is a rule D which, for every $(p, w) \in Z$ specifies exactly one finitely additive probability measure q over the class of all subsets of $B(p, w)$.

Let D be a SDF, and let $q = D(p, w)$. Then for every subset A of $B(p, w)$, $q(A)$ is interpreted as the probability that the consumer choses a bundle contained in the set A for every price-wealth situation (p, w). We denote $q' = D(p', w')$ etc.

Definition 2 (Bandyopadhyay et al. 2004) A DDF d is *tight* if, for every $(p, w) \in Z$, $p \cdot d(p, w) = w$. An SDF D is *tight* if, for every $(p, w) \in Z$, $q\left(\{x \in B(p, w) : p \cdot x = w\}\right) = 1$.

We follow the same notation as in Bandyopadhyay et al. (2004):

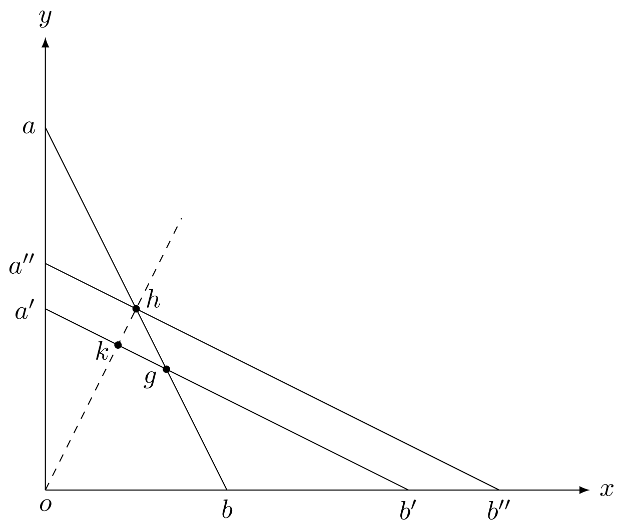

Figure 1: oab is the budget set $B(p,w)$, $oa'b'$ is the budget set $B(p',w')$, $oa''b''$ is the budget set $B(p',\lambda w')$. I is the singleton set containing g. G is ag excluding the point g. H is gb excluding the point g. G' is gb' excluding the point g. H' is $a'g$ excluding the point g. I_λ is the singleton set containing h. G_λ is ah excluding the point h. H_λ is hb excluding the point h. G'_λ is hb'' excluding the point h. H'_λ is $a''h$ excluding the point h. $I'_\lambda\left(\frac{1}{\lambda}\right)$ is the singleton set containing k. $G'_\lambda\left(\frac{1}{\lambda}\right)$ is kb' excluding the point k.

Notation 1 Given two price-wealth situations (p,w) and (p',w'), let

$$\begin{aligned}
I &= \left\{x \in \mathbb{R}^\ell_+ : p \cdot x = w \text{ and } p' \cdot x = w'\right\}\\
G &= \left\{x \in \mathbb{R}^\ell_+ : p \cdot x = w \text{ and } p' \cdot x > w'\right\}\\
H &= \left\{x \in \mathbb{R}^\ell_+ : p \cdot x = w \text{ and } p' \cdot x < w'\right\}\\
G' &= \left\{x \in \mathbb{R}^\ell_+ : p \cdot x > w \text{ and } p' \cdot x = w'\right\}\\
H' &= \left\{x \in \mathbb{R}^\ell_+ : p \cdot x < w \text{ and } p' \cdot x = w'\right\}.
\end{aligned}$$

Furthermore, we introduce the following notation:

Notation 2 Given two price-wealth situations (p,w) and (p',w'), let

$$\begin{aligned}
I_\lambda &= \left\{x \in \mathbb{R}^\ell_+ : p \cdot x = w \text{ and } p' \cdot x = \lambda\, w'\right\}\\
G_\lambda &= \left\{x \in \mathbb{R}^\ell_+ : p \cdot x = w \text{ and } p' \cdot x > \lambda\, w'\right\}\\
H_\lambda &= \left\{x \in \mathbb{R}^\ell_+ : p \cdot x = w \text{ and } p' \cdot x < \lambda\, w'\right\}\\
G'_\lambda &= \left\{x \in \mathbb{R}^\ell_+ : p \cdot x > w \text{ and } p' \cdot x = \lambda\, w'\right\}\\
H'_\lambda &= \left\{x \in \mathbb{R}^\ell_+ : p \cdot x < w \text{ and } p' \cdot x = \lambda\, w'\right\}.
\end{aligned}$$

Definition 3 Given a set $A \subset \mathbb{R}^\ell_+$ and a scalar $\lambda > 0$, $A(\lambda)$ is defined as the projection of A which is obtained by multiplying the coordinates of all elements of A by λ.

See Figure 1 for an illustration of Notations 1 and 2.

3 Stochastic Homothetic Substitution

We first recall the relevant part from the work of Bandyopadhyay et al. (1999, 2004).

Notation 3 (Bandyopadhyay et al. 1999) Let T be the set of all orderings over $\mathbb{R}^{\ell}_{+}$, the elements of T being denoted by J, J', etc. For all subsets G and H of $\mathbb{R}^{\ell}_{+}$ such that G is non-empty, let $T(G, H)$ be the set of all $J \in T$ such that J has a unique greatest element in G and this unique greatest element in G belongs to H.

Definition 4 (Bandyopadhyay et al. 1999) An SDF D satisfies *rationalizability in terms of stochastic orderings* (RSO) if there exists a probability measure r defined on the set of all subsets of T such that, for every price-wealth situation (p, w) and every subset A of $B(p, w)$, $q(A) = r[T(B(p, w), A)]$, where $q = D(p, w)$.

Definition 5 (Bandyopadhyay et al. 2004) A tight SDF D satisfies *stochastic substitutability* (SS) if for every ordered pair of price-wealth situations (p, w) and (p', w') and for all $A \subseteq I$ we have

$$q'(G') + q'(A) \geq q(H) + q(A). \tag{1}$$

Proposition 1 (Bandyopadhyay et al. 1999,Bandyopadhyay et al. 2004) For tight stochastic demand functions, rationalizability in terms of stochastic orderings implies but is not implied by stochastic substitutability.

Remark 1 Bandyopadhyay et al. (1999) show that RSO implies but is not implied by their weak axiom of stochastic revealed preference (WASRP). Bandyopadhyay et al. (2004) show that WASRP is equivalent to SS for tight SDF.

Next we introduce the relevant notations and definitions to strengthen the framework for homothetic orderings.

Definition 6 An ordering J is *homothetic* if xJy implies $\lambda x J \lambda y$ for any scalar $\lambda > 0$.

Notation 4 Let T_h be the set of all homothetic orderings over $\mathbb{R}^{\ell}_{+}$, the elements of T_h being denoted by J_h, J'_h, etc. Otherwise the notation is analogous to Notation 3.

Definition 7 An SDF D satisfies *rationalizability in terms of stochastic homothetic orderings* (RSHO) if there exists a probability measure r defined on the set of all subsets of T_h such that, for every price-wealth situation (p, w) and every subset A of $B(p, w)$, $q(A) = r[T_h(B(p, w), A)]$, where $q = D(p, w)$.

Lemma 1 Rationalizability in terms of stochastic homothetic orderings implies that for every price-wealth situation (p, w) and all $A \subseteq B(p, w)$ we have $q\,(A) = q_\lambda\,(A(\lambda))$, where $q_\lambda = D(p, \lambda\, w)$.

We omit the proof for Lemma 1 as it is obvious.

Definition 8 A tight SDF D satisfies *stochastic homothetic substitutability* (SHS) if, for every ordered pair of price-wealth situations (p, w) and (p', w') and for all $\lambda > 0$ and all $A_\lambda \subseteq I_\lambda$ we have

$$q'\left(G'_\lambda\left(\frac{1}{\lambda}\right)\right) + q'\left(A_\lambda\left(\frac{1}{\lambda}\right)\right) \geq q(H_\lambda) + q(A_\lambda). \tag{2}$$

Proposition 2 Rationalizability in terms of stochastic homothetic orderings implies but is not implied by stochastic homothetic substitutability.

Proof. Suppose a tight SDF D satisfies RSHO. RSHO implies RSO, and RSO implies SS, hence SS must be satisfied. Let $(p'', w'') = (p', \alpha\, w')$, with $\alpha > 0$. Then by SS, for the pair $\langle (p, w), (p'', w'') \rangle$ we have for every $A \subseteq I$

$$q''(G'') + q''(A) \geq q(H) + q(A), \tag{3}$$

where

$$\begin{aligned} G'' &= \left\{ x \in \mathbb{R}^{\ell}_{+} : px > w \text{ and } p''x = w'' \right\} \\ I &= \left\{ x \in \mathbb{R}^{\ell}_{+} : px = w \text{ and } p''x = w'' \right\} \\ H &= \left\{ x \in \mathbb{R}^{\ell}_{+} : px = w \text{ and } p''x < w'' \right\}. \end{aligned}$$

We have $G'' = G'_\alpha$, $H = H_\alpha$, $I = I_\alpha$, and $q'' = q'_\alpha$. Therefore (3) is equivalent to

$$q'_\alpha(G'_\alpha) + q'_\alpha(A_\alpha) \geq q(H_\alpha) + q(A_\alpha) \tag{4}$$

for every $A_\alpha \subseteq I_\alpha$. By Lemma 1 we have with RSHO that $q'_\alpha(G'_\alpha) = q'\left(G'_\alpha\left(\frac{1}{\alpha}\right)\right)$ and $q'_\alpha(A_\alpha) = q'\left(A_\alpha\left(\frac{1}{\alpha}\right)\right)$. This holds for every $\alpha = \lambda$, and we obtain SHS.

Liu and Wong (2000) showed that acyclicity of the homothetic closure of a revealed preference relation is equivalent to rationalizability in terms of deterministic homothetic preferences. Because asymmetry is a weaker condition than acyclicity, asymmetry does not imply rationalizability. To see that SHS does not imply RSHO, it can be easily checked that for a deterministic demand function, which is induced by a degenerate SDF (see Bandyopadhyay et al. 2004, Definition 2.2), SHS is equivalent to asymmetry of the homothetic closure. □

4 Discussion

Dasgupta (2005) introduced a consistency postulate for input-output choices by firms and generalized the approach to stochastic supply functions in Dasgupta (2009). An analogous expansion of the work in this paper to homothetic production functions would be a natural subject for future work.

Nandeibam (2009) examined rationalizability of a choice system in a general choice context. Where his universal set of alternatives is an affine space, we can consider homothetic utility functions. In his notation, this would imply the condition that $P^B(S) = P^{B(\lambda)}(S(\lambda))$, i.e. that the probability that the choice set from the feasible set B belongs to S (S being an element of an algebra on the set of all nonempty subsets of B) equals the probability that the choice set from $B(\lambda)$ belongs to $S(\lambda)$, where $B(\lambda)$ and $S(\lambda)$ are defined as in Definition 3. This condition, however, might already be entailed in Nandeibam's rationalizability condition (Theorem 1) if the set of allowable utility functions of an agent is restricted to homothetic utility functions. In particular, let u denote elements of the set of homothetic utility functions, then $\{u : \arg\max_B\{u(x)\} \in S\} = \{u : \arg\max_{B(\lambda)}\{u(x)\} \in S(\lambda)\}$. While Nandeibam's definition of rationalizability would not need to be strengthened for homothetic utility functions beyond the aforementioned restriction on the set of utility functions, an analysis of what additional (if any) conditions homothetic rationalizability implies remains a subject for future investigation.

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