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A simple test for the violation of the non-satiation axiom under uncertainty: The theory

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Abstract

The validity of most axioms which underlie the expected utility model has been the object of intense empirical testing. These include the independence, betweenness, transitivity, monotonicity, reduction, and non-satiation axioms. The sole, present-day test for the non-satiation axiom is predicated on the first-degree stochastic dominance theorem. This paper outlines the theory for a new, alternative test – one that is predicated on a mean-variance-preserving transformation of a one-trial binomial distribution.

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1. Introduction

The theoretical foundations for the expected utility model were laid down by Daniel Bernoulli (1738), Frank Ramsey (1931), and John von Neumann and Oscar Morgenstern (1944). A detailed history of this model can be found in Schoemaker (1980 and 1982). And overviews to empirical research into the expected utility model can be found in Kahneman and Tversky (1979), Schoemaker (1980 and 1982), Machina (1987a and 1987b), and Yaqub et al. (2009).

The validity of many axioms which underlie the expected utility model has been the object of intense empirical testing, beginning with Allais (1953). The independence, betweenness, transitivity, monotonicity, and reduction axioms have been subjected to empirical testing [Yaqub et al. (2009, p. 117)]. Levy and Levy (2001) have recently tested the validity of the non-satiation axiom, using the first-degree stochastic dominance theorem.

This paper outlines a new, alternative approach to testing for the violation of the non-satiation axiom. The conceptual origin of this test is the mean-variance-preserving transformation of a one-trial binomial distribution due to Sproule (1993), which was motivated by the notion of downside risk due to Menezes et al. (1980).

This paper is organized as follows. Section 2 offers a cursory review of three salient, background literatures. Section 3 outlines the theoretical framework for our new test of the non-satiation axiom. Final remarks are offered in Section 4.

2. Literature Review

To provide a context for the development of our new test, we offer here cursory comments on three background literatures. The first of these concerns the research by experimental psychologists into the preference orderings for simple lotteries that began more than a half a century ago. The second concerns elements of research into the first-degree stochastic dominance theorem. The third concerns elements of research into the third-degree stochastic dominance theorem.

2.1. The Research By Experimental Psychologists Into The Preference Orderings For Simple Lotteries: Beginning perhaps with Daniel Bernoulli's (1738) resolution of the St. Petersburg paradox, academics have shown a research interest in simple gambles. One possible reason for this interest is offered by Lola Lopez (1983), an experimental psychologist, who once observed: "The simple, static lottery or gamble is as indispensable to research on risk as the fruitfly to genetics. The reason is obvious: lotteries, like fruitflies, provide a simplified laboratory model of the real world, one that displays its essential characteristics while allowing for manipulation and control of important experimental variables" (p. 137).

Lopez's observation was directed at a now- or mostly-defunct research program, which was driven by experimental psychologists who endeavoured to discover the preference orderings held by their subjects for matched pairs of two- and three-outcome lotteries. This was a research program which began in the 1950s [e.g., Edwards (1953, 1954, and 1955)], and which reached its zenith in the late 1960s [e.g., Slovic and Lichtenstein (1968)].

This same research program has two main avenues of inquiry. The first explores the preference orderings by held by experimental subjects for lottery pairs that differ in the magnitude of their probabilities versus the magnitude of their payoffs. The second explores the preference orderings for lottery pairs that differ in their central moments.

This latter avenue of inquiry led to an interest in the theory of pairs of one- and two-trial binomial distributions that may differ by a mean-preserving transformation, or by a mean-variance-preserving transformation. Examples that capture the letter or the spirit of this can be found in Coombs and Pruitt (1960), van der Meer (1963), and Slovic and Lichtenstein (1968). And overviews to this literature can be found in Payne (1973), Libby and Fishburn (1977), and Schoemaker (1979).

2.2. The First-Degree Stochastic Dominance Theorem: The first-degree stochastic dominance theorem is an integral part of a research program called stochastic dominance. "Stochastic dominance is a term which refers to a set of relations that may hold between a pair of distributions" [Davidson (2008)]. In economics and finance, stochastic dominance (more often than not) takes on a narrower definition, that being, the rank ordering of pairs of distributions, when the underlying analysis is coupled with the expected utility function [Hadar and Russell (1978)]. Thus, the notions of first-, second-, and third-degree stochastic dominance arise when a particular combination of restrictions is imposed upon the distributional pairs and on the associated von Neumann-Morgenstern utility function [Elton and Gruber (1981), Levy (1992 and 1998), Levy and Weiner (1998), and Wolfstetter (1999)].

In a recent study, Levy and Levy (2001) explored the applicability of notions of first- and second-degree stochastic dominance, in explaining the choices made by students of finance between lottery pairs. In their test of the first-degree stochastic dominance theorem, Levy and Levy (2001) asked the participants to state their preference between two lotteries, X and Y, which are defined as follows:

Lottery X		***Lottery Y***	
Gain or loss	Probability	Gain or loss	Probability
-500	1/3	-500	1/2
+2500	2/3	+2500	1/2

The first-degree stochastic dominance (FSD) theorem predicts that the subjects or respondents will prefer Lottery X to Lottery Y. This prediction proved consistent with Levy and Levy's (2001) data. They wrote: (a) that "95% of the subjects selected an alternative which conforms with FSD" and (b) that "most people are rational in the sense that they prefer more rather than less money" [Levy and Levy (2001, p. 238)]. It is important to note here, that by confirming the FSD theorem, Levy and Levy's (2001) survey data also confirm the non-satiation axiom, because the non-satiation axiom is integral to the FSD theorem.

2.3. The Third-Degree Stochastic Dominance Theorem And Mean-Variance-Preserving Transformations: In motivating their development and definition of the notion of "downside risk", a notion which is tied to third-degree stochastic dominance

[Levy (1992)], Menezes et al. (1980) cited an empirical study by Mao (1970). In this study, Mao reported that: (a) the executives of eight medium and large companies were presented with a lottery pair, (b) both members of this pair had the same mean and variance, while one member was positively skewed, and the other was negatively skewed, and (c) these executives had no clear preference for either member. Mao reported that one third of his subjects chose the positively-skewed lottery, one third chose the negatively-skewed lottery, and the "remaining third indicated that their choice would depend on circumstances" (p. 349).

For purposes of the analysis in Section 3, it should be noted here: (a) that Sproule (1993) developed a general analytical framework for the lottery pair used by Mao (1970), by defining the mean-variance-preserving transformation for the one-trial binomial distribution, and (b) that Sproule's (1993) mean-variance-preserving transformation serves as the basis for the new, alternative test for the non-satiation axiom found below.

3. A New Test For The Non-Satiation Axiom

To define our test for the non-satiation axiom, we proceed as follows. First, we define two binary lotteries that differ by a mean-variance-preserving transformation. And second, we show that the relative ranking of these two lotteries by the expected utility function (hereafter EUF) depends on whether or not the von Neumann-Morgenstern utility function adheres to the non-satiation axiom.

3.1. Two Binary Lotteries, And Their Moments: Consider two binary lotteries, Lottery x and Lottery y, which are defined by three real-valued parameters, μ, α, and p, such that $0 < \alpha < \mu$ and $0 < p < 1/2$. In particular,

Assumption 1: Lottery x has as outcomes, $\mu - \alpha$ and $\mu + \alpha \dfrac{p}{1-p}$, with the probabilities of p and $1-p$ respectively.

Assumption 2: Lottery y has as outcomes, $\mu - \alpha \dfrac{p}{1-p}$ and $\mu + \alpha$, with the probabilities of $1-p$ and p respectively.

Let E(z) denote the mean, V(z) denote the variance, and S(z) denote the skewness, of Lottery z, where z = x,y. The orderings of the first three moments of Lotteries x and y are as follows:

Proposition 1 [Sproule (1993)]: If Assumptions 1 and 2 hold, then:

(a) $E(x) = E(y) = E(z) = \mu$,

(b) $V(x) = V(y) = V(z) = \alpha^2 \dfrac{p}{1-p}$, and

(c) $S(x) \;=\; -\;S(y) \;=\; -\;p.\alpha^3\left(1\;-\;\left(\frac{p}{1-p}\right)^2\right) \;<\; 0\,.$

where E denotes the expectation operator, V denotes the second central-moment, variance, and S denotes the third central-moment, skewness.

Proof: (a) $E(x) \;=\; p\left(\mu-\alpha\right) \;+\; (1-p)\left(\mu+\alpha\left(\frac{p}{1-p}\right)\right) \;=\; \mu \;-\; p\alpha \;+\; p\alpha \;=\; \mu$

$$E(y) \;=\; (1-p)\left(\mu-\alpha\left(\frac{p}{1-p}\right)\right) \;+\; p\left(\mu+\alpha\right) \;=\; \mu \;-\; p\alpha \;+\; p\alpha \;=\; \mu$$

(b) $V(x) \;=\; E\left(x-\mu\right)^2$

$$= \; p\left(\mu-\alpha-\mu\right)^2 \;+\; (1-p)\left(\mu+\alpha\left(\frac{p}{1-p}\right)-\mu\right)^2$$

$$= \; p\alpha^2 \;+\; (1-p)\left(\frac{\alpha p}{1-p}\right)^2 \;=\; p\alpha^2 \;+\; \frac{(\alpha p)^2}{1-p}$$

$$= \; p\alpha^2\left(1+\frac{p}{1-p}\right) \;=\; p\alpha^2\,\frac{1-p+p}{1-p} \;=\; \alpha^2\,\frac{p}{1-p}$$

$V(y) \;=\; E\left(y-\mu\right)^2$

$$= \; (1-p)\left(\mu-\alpha\left(\frac{p}{1-p}\right)-\mu\right)^2 \;+\; p\left(\mu+\alpha-\mu\right)^2$$

$$= \; (1-p)\alpha^2\left(\frac{p}{1-p}\right)^2 \;+\; p\alpha^2 \;=\; \alpha^2\,\frac{p}{1-p}$$

(c) $S(x) \;=\; E\left(x-\mu\right)^3$

$$= \; p\left(\mu-\alpha-\mu\right)^3 \;+\; (1-p)\left(\mu+\alpha\left(\frac{p}{1-p}\right)-\mu\right)^3$$

$$= \; -\;p\alpha^3 \;+\; (1-p)\left(\frac{\alpha p}{1-p}\right)^3 \;=\; -\;p\alpha^3 \;+\; \frac{(\alpha p)^3}{\left(1-p\right)^2}$$

$$= \; -\;p\alpha^3\left(1\;-\;\left(\frac{p}{1-p}\right)^2\right) \;<\; 0$$

$S(y) \;=\; E\left(y-\mu\right)^3$

$$= \; (1-p)\left(\mu-\alpha\left(\frac{p}{1-p}\right)-\mu\right)^3 \;+\; p\left(\mu+\alpha-\mu\right)^3$$

$$= (1-p)\left(-\alpha\left(\frac{p}{1-p}\right)\right)^3 + p\alpha^3 = \frac{(-\alpha p)^3}{1-p} + p\alpha^3$$

$$= p\alpha^3\left(1 - \left(\frac{p}{1-p}\right)^2\right) > 0. \qquad \bullet$$

3.2. A Preference Ordering Of Lotteries x and y: Let $U(z)$ denote the von Neumann-Morgenstern utility function, given z. By Assumption 1, the EUF for Lottery x is defined as,

$$E[U(x)] = pU(\mu-\alpha) + (1-p)U\left(\mu+\alpha\left(\frac{p}{1-p}\right)\right) \tag{1}$$

and by Assumption 2, the EUF for Lottery y is defined as

$$E[U(y)] = (1-p)U\left(\mu-\alpha\left(\frac{p}{1-p}\right)\right) + pU(\mu+\alpha) \tag{2}$$

We have shown that Lotteries x and y differ by their third central-moment [Proposition 1]. What remains to be shown is that the relative ranking of Lotteries x and y by the EUF depends on whether or not the non-satiation axiom is satisfied. Stated differently, we will show next that if and only if the von Neumann-Morgenstern utility function satisfies the non-satiation axiom, then the EUF ranks Lottery y over Lottery x. In particular,

Assumption 3 [Non-Satiation]: $U^{(1)}(z) > 0$ for all z.

Assumptions 1, 2, and 3 serve to define the preference ordering based on the EUF.

Proposition 2: If Assumptions 1, 2, and 3 hold, then $E[U(x)] - E[U(y)] < 0$.

Proof: By Assumptions 1 and 2,

$$\mu-\alpha < \mu-\alpha\left(\frac{p}{1-p}\right) < \mu+\alpha\left(\frac{p}{1-p}\right) < \mu+\alpha \tag{3}$$

By Assumption 3, Equation (3) may be rewritten as:

$$U(\mu-\alpha) < U\left(\mu-\alpha\left(\frac{p}{1-p}\right)\right) < U\left(\mu+\alpha\left(\frac{p}{1-p}\right)\right) < U(\mu+\alpha)$$

$$\Leftrightarrow U(\mu+\alpha) - U(\mu-\alpha) > U\left(\mu+\alpha\left(\frac{p}{1-p}\right)\right) - U\left(\mu-\alpha\left(\frac{p}{1-p}\right)\right)$$

$$\Leftrightarrow U(\mu-\alpha) - U(\mu+\alpha) < U\left(\mu-\alpha\left(\frac{p}{1-p}\right)\right) - U\left(\mu+\alpha\left(\frac{p}{1-p}\right)\right)$$

$$\Leftrightarrow \; p\big(U(\mu-\alpha) \; - \; U(\mu+\alpha)\big)$$

$$< \; (1-p)\left(U\left(\mu-\alpha\left(\frac{p}{1-p}\right)\right) \; - \; U\left(\mu+\alpha\left(\frac{p}{1-p}\right)\right)\right)$$

$$\Leftrightarrow \; pU(\mu-\alpha) \; - \; pU(\mu+\alpha)$$

$$< \; (1-p)U\left(\mu-\alpha\left(\frac{p}{1-p}\right)\right) \; - \; (1-p)U\left(\mu+\alpha\left(\frac{p}{1-p}\right)\right)$$

$$\Leftrightarrow \; pU(\mu-\alpha) \; + \; (1-p)U\left(\mu+\alpha\left(\frac{p}{1-p}\right)\right)$$

$$- \left((1-p)U\left(\mu-\alpha\left(\frac{p}{1-p}\right)\right) \; + \; pU(\mu+\alpha)\right) \; < \; 0$$

$$\Leftrightarrow \; E[U(x)] \; - \; E[U(y)] \; < \; 0 . \qquad \bullet$$

3.3. Our New Test in Summary Form: To test the non-satiation axiom, the experimenter should present to the subject the choice between two lotteries, Lottery x and Lottery y (as defined by Propositions 1 and 2 above). If the subject prefers Lottery y over x, then he or she is said to satisfy the non-satiation axiom. But if the subject prefers Lottery x over Lottery y, or if the subject is indifferent between Lottery x and Lottery y, then he or she is said to violate the non-satiation axiom.

4. Final Remarks

The validity of most axioms which underlie the expected utility model has been the object of intense empirical testing. These include the independence, betweenness, transitivity, monotonicity, reduction, and non-satiation axioms. One test for the non-satiation axiom is provided by the first-degree stochastic dominance theorem. This paper provided a second test, by virtue of a mean-variance-preserving transformation of a one-trial binomial distribution.

In Section 2, we offered a cursory review of three literatures that serve as the basis or foundation of the present research. And in Section 3, we presented our new test for the non-satiation axiom.

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