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Stability under learning: the neo-classical growth problem

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Abstract

A local stability condition for the standard neo-classical Ramsey growth model is derived. The proposed setting is deterministic, defined in discrete time and expectations are formed through adaptive learning. The stability condition imposes an upper bound on the long-term value of the gain sequence.

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1 Introduction

Modern macroeconomics have progressively replaced the notion that agents are fully rational by a concept of learning, under which individuals and firms collect information over time, learn with this information and eventually accomplish a long-term capacity to produce optimal decisions. Rational expectations / perfect foresight emerge, then, more reasonably as a long-run possibility and not as an every period underlying assumption. The literature on macroeconomic learning is extensive and covers almost all the relevant phenomena, like monetary policy, asset pricing or growth, and the analysis is undertaken both in stochastic and deterministic environments.

Our concern in this paper will be with deterministic economic growth and the aim is to derive a simple and straightforward condition of stability for the neo-classical Ramsey growth model. Deterministic models of growth under learning have been analyzed before in the literature, but essentially in the context of overlapping generations models, e.g. by Bullard (1994) Sorger (1998) and Schonhofer (1999). A relevant exception is Cellarier (2006) who effectively considers the intertemporal utility maximization setup; however, the concern of this author is essentially with the search for endogenous business cycles in a scenario where convergence to rational expectations is excluded from the start.

Other studies that focus on the implications of learning in the context of simple neo-classical growth models or real business cycle (RBC) models have concentrated essentially on the impact of stochastic disturbances over equilibrium results and transitional paths. This is the case of Huang, Liu and Zha (2009), who assume an RBC environment and conclude that the long-term equilibrium is the same whether expectations are formed under adaptive learning or, alternatively, expectations are purely rational; the difference is found on the transitional dynamics in the vicinity of the steady-state, which are substantially different from one expectations' formation assumption to the other. The main result of this study is that adaptive learning may constitute an important source of frictions that tend to amplify and propagate technology shocks. The referred study is a reply to Williams (2003), who encounters no relevant differences between learning dynamics and rational expectations dynamics in RBC environments, in the sense that learning does not influence optimizing decisions of the economic agents.

Considering, as well, a real business cycle model, Eusepi and Preston (2008) also address the implications of learning over the propagation of technology shocks. The main finding in that paper is that shifting expectations can be interpreted as a source of business cycle fluctuations; more precisely, if households and firms have an incomplete knowledge about the environment that surrounds them, they will have to form beliefs and the economy becomes self-referential, i.e., shifts in beliefs about future returns to both labor and capital will have impact in current prices which, in turn, may reinforce previous beliefs. As a result, current prices can lose their capacity to inform about future economic conditions and this can constitute a source of instability that tends to generate and propagate fluctuations in real economic activity. In short, the argument is that learning and the shifts in expectations it generates tend to amplify the volatility of economic series making business cycles to become self-fulfilling. An RBC model where rational expectations are replaced by some sort of learning mechanism is a model better equipped to explain large volatility episodes (in particular, learning amplifies technology shocks).

In Carceles-Poveda and Giannitsarou (2007) we encounter another relevant contribution relating the role of learning in general equilibrium stochastic models. In this specific case, an example of a growth model is presented (the labor-leisure choices are ignored and the RBC environment gives place to a simple stochastic Ramsey growth model). Again, as in the above cited studies, a central role is attributed to technology shocks; these will be decisive on how learning influences the time paths of the endogenous variables. The main findings, directly attached to the existence of a random disturbance, indicate that the behavior of aggregate variables depends not only on the selected learning algorithm but also on the initial state of the system.

The present paper departs from the cited references because it concentrates on a deterministic version of the simple neo-classical growth model and considers a constant gain learning algorithm in order to search for a local dynamics stability condition. By taking the educated guess that consumption grows at a constant long-term growth rate, the representative agent will consider a perceived law of motion that allows to estimate the value of such growth rate. The mechanism through which the estimation is produced consists on a simple regression using ordinary least squares. A similar study is undertaken in Gomes (2009) for an endogenous growth problem (where the production function exhibits constant marginal returns concerning the accumulation of capital), rather than a neo-classical environment characterized by decreasing marginal returns on the accumulation of reproducible inputs. In that paper, as in the analysis to develop in this note, the main result relates to a threshold effect that is found respecting the quality of the learning process: only a relatively efficient learning process will allow for stability of the long-term steady-state.

The relevant parameter in the analysis will be the steady-state level of the gain sequence. This parameter indicates whether the learning process was successful (in the sense that it allows for asymptotic perfect foresight) or not. Optimal or efficient learning requires the gain sequence to converge to zero; otherwise, if it remains at any value between 0 and 1 the learning process is not efficient (the more it departs from 0, the larger is the degree of inefficiency); the mentioned efficiency concept is related to the ability of agents in avoiding incurring in systematic mistakes and therefore in producing expectations that, under a fully deterministic environment, are perfect (are correct with probability 1).

However, the absence of a perfect process of learning must not be interpreted as an uncommon or even an undesired outcome; taking the words of Sobel (2000), 'Agents in these models begin with a limited understanding of the problems that they must solve. Experience improves their decisions. Death and a changing environment worsen them.' (page 241), and, furthermore, 'An agent will not necessarily learn the optimal decision when the cost of acquiring additional information exceeds the benefits.' (page 244).

An important feature of many learning mechanisms, as the adaptive learning setup we consider, is that agents do not necessarily need to accomplish the rational expectations long-term outcome to generate exactly the same steady-state result as if they did. Rather, there is generally a minimal requirement in terms of the long-term capacity of predicting future values that produces precisely the same result as under perfect foresight. If learning is costly (and, effectively, there are always costs in acquiring and processing information), then the effort on reaching an optimal forecasting capacity does not pay; the agent benefits in locating at the point in which: (*i*) stability at the perfect foresight level of the considered endogenous variables holds; (*ii*) the costs of learning are the lowest possible. Below, we derive a straightforward condition for stability that reveals that the higher is the level of technology and the lower are the discount rate of future utility and the depreciation rate of capital, the less the representative agent will need to learn in order to accomplish the intended long-term result.

The remainder of this note is organized as follows. Section 2 presents the structure of the growth model and introduces the adaptive learning mechanism. Section 3 explains how to transform the model into a linearized system in the neighborhood of the steady-state, allowing for a local stability analysis. The stability condition is derived in section 4. Section 5 concludes.

2 The Growth Model and the Learning Mechanism

Consider a standard one-sector optimal growth model. A representative agent maximizes consumption utility intertemporally, under an infinite horizon and taking a positive future utility discount rate, ρ. Thus, the agent maximizes $V_0 = \sum_{t=0}^{+\infty} \left(\frac{1}{1+\rho}\right)^t U(c_t)$, with $U(c_t) : \mathbb{R}_+ \to \mathbb{R}$ the instantaneous utility function; variable c_t represents per capita consumption. The utility function must obey to trivial conditions of continuity and differentiability, and marginal utility must be positive and diminishing. To aid on the tractability of the model, we assume a simple logarithmic utility function $U(c_t) = \ln c_t$.

The resource constraint of the problem is the conventional capital accumulation equation: $k_{t+1} = f(k_t) - c_t + (1-\delta)k_t$, k_0 given. Variable $k_t \geq 0$ represents the per capita stock of capital and $\delta > 0$ refers to the rate of capital depreciation. The production function is neo-classical, i.e., it evidences decreasing marginal returns. Assuming a Cobb-Douglas production technology, we consider $f(k_t) = Ak_t^\alpha$, with $A > 0$ the technology index and $\alpha \in (0,1)$ the output-capital elasticity.

Maximizing V_0 subject to the resource constraint, one derives three first-order conditions:

$$E_t p_{t+1} = 1/c_t;$$

$$[1 + \alpha A k_t^{-(1-\alpha)} - \delta] E_t p_{t+1} = (1+\rho) p_t;$$

$$\lim_{t \to +\infty} k_t \left(\frac{1}{1+\rho}\right)^t p_t = 0 \text{ (transversality condition).}$$

In these conditions, p_t stands for the shadow-price of capital and $E_t p_{t+1}$ is the expected value of the shadow-price for the subsequent time period. From the first optimality condition, we infer that $E_{t+1} p_{t+2} = 1/E_t c_{t+1}$, and therefore we resort to the second optimality condition to present an equation of motion for the next period expected per capita consumption level,

$$E_t c_{t+1} = \frac{1 + \alpha A k_{t+1}^{-(1-\alpha)} - \delta}{1+\rho} c_t \quad (1)$$

The perfect foresight steady-state for the system composed by the capital constraint and equation (1) is obtained by imposing $\overline{k} := k_{t+1} = k_t$ and $\overline{c} := E_t c_{t+1} = c_{t+1} = c_t$. Straightforward computation conducts to the unique steady-state pair of values

$$(\overline{k}, \overline{c}) = \left[\left(\frac{\alpha A}{\rho + \delta}\right)^{1/(1-\alpha)}; \frac{1}{\alpha}(\rho + (1-\alpha)\delta)\overline{k}\right].$$

Under perfect foresight, the system is saddle-path stable, i.e., if the one-dimensional stable path is followed, the convergence towards the steady-state point is fulfilled.

Assume that expectations about the next period level of consumption are formed through adaptive learning. Following related literature [e.g., Bullard (1994) or Adam, Marcet and Nicolini (2008)], we consider an estimator variable b_t such that $E_t c_{t+1} = b_t c_t$. The estimator is updated taking into account past information and using the rule $b_t = b_{t-1} + \sigma_t \left(\frac{c_{t-1}}{c_{t-2}} - b_{t-1} \right)$, b_0 given. Variable $\sigma_t \in (0,1)$ respects to the gain sequence, as characterized in the introduction. We do not need to explicitly model the time evolution of this variable because we will concentrate the analysis in the long-run properties of the growth system. It is simply necessary to know that if σ_t converges to zero ($\overline{\sigma} = 0$), a steady-state perfect foresight result is attained (i.e., the unique steady-state point is accomplished; in this point, consumption takes a constant value and, thus, $b_t = 0$), while if σ_t converges to any positive value lower than 1, then a less than optimal long-run forecasting ability is evidenced (the higher is $\overline{\sigma}$, the lower will be the steady-state quality of the forecasts).

3 Linear Approximation in the Vicinity of the Steady-State

The goal is to analyze local stability conditions, i.e., conditions under which convergence to $(\overline{k}, \overline{c})$ is accomplished, for a given pair (k_0, c_0) close to equilibrium. Working in the neighborhood of the steady-state point, we linearize the system of difference equations relating to the motion of capital and consumption in order to attain stability conditions.

The linearization procedure is undertaken in two steps. First, we linearize function $F(k_t, c_t) := E_t c_{t+1}/c_t$; this allows to write the estimator as a linear function of the two endogenous variables, opening the way for explicitly presenting a system of capital-consumption equations defined in terms of contemporaneous and past values of variables. Second, we linearize the obtained system in order to build a Jacobian matrix from which stability conditions are straightforward to derive.

Given the relation between expected consumption, present consumption and the estimator, in the neighborhood of the steady-state we can write: $b_t \simeq 1 + F_k(\overline{k}, \overline{c})(k_t - \overline{k}) + F_c(\overline{k}, \overline{c})(c_t - \overline{c})$. Straightforward computation allows to find $F_k(\overline{k}, \overline{c}) = -(1-\alpha)(\rho+\delta)/\overline{k}$ and $F_c(\overline{k}, \overline{c}) = (1-\alpha)(\rho+\delta)/((1+\rho)\overline{k})$. Therefore, defining $\theta := 1 + \frac{\alpha-(1-\alpha)(\rho+\delta)}{\alpha(1+\rho)}(1-\alpha)(\rho+\delta)$, one arrives to $b_t \approx \theta - (1-\alpha)(\rho+\delta)\frac{k_t}{\overline{k}} + (1-\alpha)\frac{(\rho+\delta)}{(1+\rho)}\frac{c_t}{\overline{k}}$. Replacing this expression in the updating estimator rule, the following difference equation for consumption is obtained,

$$c_t \simeq (1-\sigma_t)[c_{t-1} - (1+\rho)k_{t-1}] + \sigma_t(c_{t-1}/z_{t-1} - \theta) + (1+\rho)k_t; \;\; z_t = c_{t-1} \tag{2}$$

The second step of the linearization procedure consists in taking the capital equation and the pair of equations (2) and evaluating them in the neighborhood of the steady-state. A three dimensional matricial system emerges,

$$\begin{bmatrix} k_t - \overline{k} \\ c_t - \overline{c} \\ z_t - \overline{c} \end{bmatrix} \simeq$$

$$\begin{bmatrix} 1+\rho & -1 & 0 \\ (1+\rho)(\rho+\overline{\sigma}) & (1-\overline{\sigma})+\overline{\sigma}/\overline{c}-(1+\rho) & -\overline{\sigma}/\overline{c} \\ 0 & 1 & 0 \end{bmatrix} \cdot \begin{bmatrix} k_{t-1} - \overline{k} \\ c_{t-1} - \overline{c} \\ z_{t-1} - \overline{c} \end{bmatrix} \quad (3)$$

4 Stability Condition

Let J be the 3×3 Jacobian matrix in (3). For this matrix, it is straightforward to compute determinant, sum of principle minors and trace. They are all positive values: $Det(J) = (1+\rho)\frac{\overline{\sigma}}{\overline{c}} > 0$; $\Sigma M(J) = (2+\rho)\frac{\overline{\sigma}}{\overline{c}} > 0$; $Tr(J) = (1-\overline{\sigma}) + \frac{\overline{\sigma}}{\overline{c}} > 0$.

Stability conditions involving determinant, sum of principle minors and trace of a three-dimensional linearized system are the following [see Brooks (2004)]:

(i) $1 - Det(J) > 0$;
(ii) $1 - \Sigma M(J) + Tr(J)Det(J) - (Det(J))^2 > 0$;
(iii) $1 - Tr(J) + \Sigma M(J) - Det(J) > 0$;
(iv) $1 + Tr(J) + \Sigma M(J) + Det(J) > 0$.

In the specific case under analysis, we observe that $\Sigma M(J) = Tr(J) + Det(J) - (1-\overline{\sigma})$. Thus, the stability conditions are reduced to:

(i) $1 - Det(J) > 0$;
(ii) $2 - \overline{\sigma} - Tr(J) - Det(J) + Tr(J)Det(J) - (Det(J))^2 > 0$;
(iii) $\overline{\sigma} > 0$;
(iv) $\overline{\sigma} + 2Tr(J) + 2Det(J) > 0$.

Conditions (iii) and (iv) are verified for any values of parameters obeying the imposed constraints. Condition (i) requires $\overline{\sigma} < \overline{c}/(1+\rho)$ and condition (ii) implies $-\frac{1+\sqrt{1+4(1+\rho)(\overline{c}+\rho)}}{2(\overline{c}+\rho)}\frac{\overline{c}}{1+\rho} < \overline{\sigma} < \frac{\sqrt{1+4(1+\rho)(\overline{c}+\rho)}-1}{2(\overline{c}+\rho)}\frac{\overline{c}}{1+\rho}$. Let $\phi := \frac{\sqrt{1+4(1+\rho)(\overline{c}+\rho)}-1}{2(\overline{c}+\rho)}$. Condition (i) will be more restrictive than condition (ii) if $\phi > 1$. This last inequality would imply $\overline{c} < 0$, which is not a feasible outcome. Therefore, the first condition can be set aside and, hence, the unique relevant stability condition is the upper bound of (ii) (note that the lower bound is below zero, and consequently it can be ignored). This result is presented in the form of a proposition,

Proposition 1 *In the neo-classical Ramsey growth model with expectations generated through adaptive learning, stability holds under condition* $\overline{\sigma} < \phi\overline{c}/(1+\rho)$, *with* $0 < \phi < 1$.

The result in proposition 1 is intuitive. It sets a boundary on learning inefficiency or, in other words, it presents a minimum requirement in terms of information acquisition and processing needed in order for the steady-state to be accomplished. As discussed in the introduction, assuming a costly learning process, the representative agent does not need to employ resources to attain $\overline{\sigma} = 0$. She just has to apply a level of effort that is enough to guarantee that $\overline{\sigma}$ is close to, but below, $\phi\overline{c}/(1+\rho)$.

Proposition 2 briefly states the determinants of the learning boundary.

Proposition 2 *The learning requirements are relaxed (i.e., the representative agent has to make less learning effort in order to reach the steady-state result) with a relatively higher level of technology and with lower depreciation and discount rates.*

The results in proposition 2 follow directly from observing that $\partial\bar{c}/\partial A > 0$, $\partial\bar{c}/\partial\delta < 0$ and $\partial\bar{c}/\partial\rho < 0$ (and noticing that $\partial\phi/\partial\bar{c} > 0$).

To close the analysis, a numerical example is presented. The benchmark values of parameters are $\alpha = 0.3$, $\delta = 0.05$ (per year), $\rho = 0.02$ (per year).[1] Parameter A is chosen to guarantee $\bar{k} = 1$, i.e., $A = 0.233$. In this case, $\bar{c} = 0.183$ and the stability condition is $\bar{\sigma} < 0.156$. The gain sequence must possess a steady-state value lower than 0.156 in order to allow for stability / convergence to the steady-state pair $(\bar{k}, \bar{c}) = (1, 0.183)$.

Results in proposition 2 can be illustrated by varying some of the parameter values. In table 1, various experiments are displayed.

Parameter values*	$\bar{k}$	$\bar{c}$	Stability condition
$\delta = 0.02$	2.220	0.252	$\bar{\sigma} < 0.205$
$\delta = 0.1$	0.462	0.139	$\bar{\sigma} < 0.122$
$\rho = 0.01$	1.244	0.187	$\bar{\sigma} < 0.160$
$\rho = 0.05$	0.600	0.170	$\bar{\sigma} < 0.142$
$A = 0.2$	0.802	0.091	$\bar{\sigma} < 0.082$
$A = 0.5$	2.971	0.545	$\bar{\sigma} < 0.387$

Table 1 - Stability condition for different values of parameters
(*The other parameters maintain the proposed benchmark values).

The stability conditions in the table confirm the results in proposition 2: to attain stability, learning becomes more demanding when the depreciation rate of capital is higher, the discount rate of future utility is higher and the level of technology regresses.

5 Final Remarks

This note has derived an explicit, simple and intuitive stability condition for the conventional Ramsey growth model when expectations about future consumption are formed through adaptive learning. The relevance of the result is that the representative agent may be boundedly rational (i.e., she may not be able to treat information with the efficiency needed in order to achieve a long-run optimal forecasting capability), and still be able to attain the intended long-run locus (the unique steady-state point). Nevertheless, there is a clear boundary: after some threshold value of learning inefficiency, stability is lost. A high technological capacity and low capital depreciation and intertemporal discount rates allow to relax the learning constraint.

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