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The timing of information updates: a stability result

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Abstract

We assume an environment where the current value of an aggregate nonstationary variable is generated by weighting the behavior of a large set of agents who choose to form expectations resorting to more or less outdated information concerning the state of the economy. Agents using recent information are able to produce expectations with a strong component of perfect foresight; agents resorting to outdated information will use predominantly the time series of the assumed variable to learn its long-term value. The main result is that a strong degree of information stickiness may imply a departure from stability (a Neimark-Sacker bifurcation occurs).

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1 Introduction

Following the literature on learning equilibria [Bullard (1994), Schonhofer (1999), Tuinstra and Wagener (2007)], we address the dynamics of a simple deterministic model involving expectations. A large set of agents is considered, all possessing knowledge about past realizations of the assumed endogenous variable; this allows them to learn, resorting to a least squares estimation, the expected long-term growth rate of the variable (which is assumed to be non-stationary).

Besides the knowledge on the time series of the aggregate, agents have access to additional information on the state of the economy; at this level we find heterogeneity: each agent updates this second set of information only sporadically, at some random time moments. If this information is recent, it allows to form expectations with a strong component of perfect foresight; if such information is outdated, then agents cannot rely on it to make an accurate forecast on the current value of the variable and, therefore, they will instead resort to the time series of the endogenous variable to learn its steady state growth rate (the present moment is seen as the long-run, a state that the agent knows to be characterized by a constant growth rate, although the value of this rate is not known, and hence it has to be learned).

The information updating setup will be similar to the sticky-information framework used in Mankiw and Reis (2002, 2006, 2007) to study agents' inattentiveness in monetary policy settings. A given parameter, representing the share of agents that in some moment update their information set, will translate the degree of information stickiness.[1]

The relevant presented result is that combining inattentiveness and least squares learning allows to encounter a Neimark-Sacker bifurcation separating, in the parameters space, a region where stability holds from an area where local instability is evidenced. A global inspection of the dynamic properties in the unstable area indicates that endogenous fluctuations are likely to arise after the bifurcation line is crossed. Given the chaotic nature of such fluctuations, one may expect them to persist if one accounts for the self-fulfilling mistake argument of Grandmont (1998).[2] Loss of stability will require a high degree of information stickiness, i.e., a small share of agents updating their information set at each time moment.

The remainder of this note is organized as follows. Section 2 characterizes the way in which information stickiness changes an otherwise simple constant growth setting. Section 3 introduces the least squares regression and presents the derived dynamic system. Section 4 highlights the main dynamic results and, finally, section 5 discusses the implications of the analyzed setup.

2 The Sticky-Information Framework

Assume an economy where agents need to know the current value of some non-stationary variable $x \in \mathbb{R}_+$. Under perfect foresight, this variable will grow over time at a rate $\gamma > 0$.

To generate a forecast on x_t, each agent has two sets of information: (i) every agent possesses knowledge about past values of x: $x_{t-1}, x_{t-2}, ...$; (ii) agents have other information on the state

[1] It is important to clarify that, in the development of the analysis, the Mankiw-Reis framework will simply provide the notion of information stickiness. We then depart from such framework by assuming that expectations are not, in every circumstance, formed under an environment of perfect foresight / rational expectations (this environment constitutes the setting under which all the results of the Mankiw-Reis monetary policy setup are built upon).

[2] We refer to this argument in section 5.

of the economy that helps on predicting the true value of x at time t. Regarding this second information set, heterogeneity on the timing of information updating is assumed; an agent that last updated her information set j periods ago, will form an expectation that in terms of the knowledge about the economic environment goes back to $t-j$. We denote this expectation by $E_t^{t-j}(x_t)$. It should be emphasized that $E_t^{t-j}(x_t)$ is the evaluation made at the beginning of period t concerning the current value of x, when the agent knows all past realizations of x and she has gathered, j periods ago, information concerning the forces governing the evolution of the variable. One may think of information that is easy to process (the observed time series of the variable) and information that requires additional effort to obtain, and therefore is collected only sporadically in time.

In this information stickiness scenario, at each past time period, a share of agents $\lambda \in [0,1]$ collects information on the state of the economy and forms expectations accordingly (i.e., using such past information). The closer the value of λ is to 1, the more flexible or less sluggish is information dissemination. Reis (2006) demonstrates that if the economy is populated by many agents then the distribution of information updating converges to a Poisson distribution. Such finding implies that each agent has an equal probability of being one of the agents updating information at a given moment, independently of the timing of the last update.

Under the Poisson process, the weighted average of the expectations of each agent, which constitutes the truly observed value of the variable at t, will be

$$\widetilde{x}_t = \lambda \sum_{j=0}^{\infty} (1-\lambda)^j E_t^{t-j}(x_t) \tag{1}$$

Two extreme situations are admissible. First, agents will update their information today. This implies that they have full knowledge about the current state of the economy and, thus, they know that the value of x is effectively x_t (there is perfect foresight): $E_t^t(x_t) = x_t$. On the other extreme, agents have updated their information set infinitely far in the past and consequently they have no knowledge on the current state of the economy except for the observed time series of x through $t-1$. Thus, they will attempt to learn the long-term growth rate of x, which is assumed constant. We have initially stated that the variable should grow over time at rate γ, if perfect foresight prevails. Thus, agents will believe that the true dynamic process is such that the system converges in the long-run to a constant growth rate result; however, they do not know the value of this rate and therefore they will estimate it. The adopted perceived law of motion (PLM) is $\widehat{x}_{t+1} = \beta_t \widehat{x}_t$, with $\widehat{x}_{t+1}/\widehat{x}_t$ the expected gross rate of growth of variable x; β is the parameter to estimate (convergence to the perfect foresight equilibrium implies $\beta = 1+\gamma$). The referred second limit case is, thus, $\lim_{J\to\infty} E_t^{t-J}(x_t) = \widehat{x}_t$.

Between the two assumed limit cases, one might conceive a continuum of possibilities, where the more j is close to zero the more forecasts approach perfect foresight and where, on the contrary, the higher is the value of j the stronger will be the influence of learning. The equation that follows intends to translate this reasoning,

$$E_t^{t-j}(x_t) = x_t^{1/(1+j)} \widehat{x}_t^{j/(1+j)} \tag{2}$$

Expression (2) attributes different weights to a perfect foresight component and to a learning component in the prediction of x, given the time moment in which information about the state of the economy was last collected.

By noticing that

$$\sum_{j=0}^{\infty}\left((1-\lambda)\left(\frac{\widehat{x}_t}{x_t}\right)^{1/(1+j)}\right)^{j} \simeq \left(\frac{\widehat{x}_t}{x_t}\right)^{1-\lambda}\Big/\lambda \tag{3}$$

we can write expression (1) as

$$\widetilde{x}_t \simeq x_t^{\lambda}\widehat{x}_t^{1-\lambda} \tag{4}$$

The displayed approximation, (3), is a good approximation under the assumption that $\widehat{x}_t$ does not depart significantly from x_t. In fact, if $\widehat{x}_t = x_t$, then $\sum_{j=0}^{\infty}(1-\lambda)^j = \frac{1}{\lambda}$. For values of $\widehat{x}_t$ different from x_t, the quality of the approximation falls as the values of the two variables diverge. Consider, as an example, the cases where $\widehat{x}_t$ is 1% above and 1% below x_t (and let $\lambda = 0.75$; any other value of λ conducts to the same type of results); in these cases we obtain, respectively,

i) $\left(\frac{\widehat{x}_t}{x_t}\right)^{1-\lambda}\Big/\lambda = 1.3366$; $\sum_{j=0}^{\infty}\left((1-\lambda)\left(\frac{\widehat{x}_t}{x_t}\right)^{1/(1+j)}\right)^{j} = 1.3352$.

ii) $\left(\frac{\widehat{x}_t}{x_t}\right)^{1-\lambda}\Big/\lambda = 1.3300$; $\sum_{j=0}^{\infty}\left((1-\lambda)\left(\frac{\widehat{x}_t}{x_t}\right)^{1/(1+j)}\right)^{j} = 1.3315$.

We observe that if $\widehat{x}_t$ and x_t possess values close to each other, equation (3) provides a reasonable approximation. Thus, (3) arises under the assumption that the considered agents are able to generate forecasts on x_t through learning that are not too divergent from the observed value. Since the learning mechanism allows $\widehat{x}_t$ to approach x_t as additional data is collected, then the quality of the approximation is guaranteed by assuming that the initial value $\widehat{x}_0$ is located in the vicinity of x_0.

Equation (4) reveals that the degree of information stickiness is vital in determining the observed value of x (i.e., $\widetilde{x}$). Complete information flexibility ($\lambda = 1$) is synonymous of perfect foresight / rational expectations; complete information stickiness ($\lambda = 0$) implies that no information exists besides the knowledge on the past behavior of x, and therefore only learning about the growth rate of x matters. The reasonable assumption that some intermediate degree of information stickiness exists implies that the observed value of the variable is a weighted geometrical average of a perfect foresight forecast and of a value obtained through learning; the weights relate to the extent of inattentiveness.

Because the true observed value of the considered variable is $\widetilde{x}_t$, it will be this value that will presumably grow at rate γ in the steady state; thus, we should take $\widetilde{x}_{t+1} = (1+\gamma)\widetilde{x}_t$ as the actual law of motion (in the long-run). This is equivalent to (5), given (4),

$$\frac{x_t}{x_{t-1}} \simeq (1+\gamma)^{1/\lambda}\left(\frac{\widehat{x}_{t-1}}{\widehat{x}_t}\right)^{(1-\lambda)/\lambda} \tag{5}$$

Note that, at the beginning of this section, one has stated that under full perfect foresight, variable x_t grows at rate γ; at this stage, we realize that x_t will not in fact grow at this rate because only a share of agents form expectations under perfect foresight. Thus, if it is the aggregate value of x (i.e., $\widetilde{x}$) that grows at rate γ, according to equation (4) variable x_t will grow at rate $\left(\frac{1+\gamma}{\beta_t^{1-\lambda}}\right)^{1/\lambda} - 1$; this rate will effectively be equal to γ in the long-run, if the rational expectations equilibrium is attained (i.e., if all agents form rational expectations in the steady-state). As long as a share of the existing agents forms expectations through learning, the growth rate of x_t simply adjusts in order to guarantee that the observed level of the variable grows at the specified rate.

In the next section, we present the learning mechanism.

3 The Adaptive Learning Mechanism

We consider least squares learning. The agents will collect data on the time series x_{t-1}, x_{t-2}, ... and then run a regression to estimate β. Agents believe in a constant β but they are unaware about its true value. The estimate is updated at each time moment as new information on the value of x becomes available.

The regression yields

$$\beta_t = \left[\sum_{s=1}^{t-1} x_{s-1}^2\right]^{-1} \cdot \left[\sum_{s=1}^{t-1} x_{s-1} x_s\right] \tag{6}$$

Replacing the estimated value into (5), one obtains

$$\frac{x_t}{x_{t-1}} \simeq (1+\gamma)^{1/\lambda} \beta_{t-1}^{-(1-\lambda)/\lambda} \tag{7}$$

Equations (6) and (7) form an expectations feedback system, where the observed growth rate of x influences the expected growth rate and this also exerts influence over the actual growth rate.

To present the system in recursive form, as a three-dimensional set of first-order difference equations, one needs to define the gain sequence, $\sigma_t := x_{t-1}^2 \cdot \left[\sum_{s=1}^{t-1} x_s^2\right]^{-1}$. The system comes

$$\begin{cases} \beta_t \simeq \beta_{t-1} + \sigma_{t-1}(\kappa_{t-1} - \beta_{t-1}) \\ z_t = \beta_{t-1} \\ \sigma_t = \left(\kappa_{t-1}^2 \sigma_{t-1}\right) / (1 + \kappa_{t-1}^2 \sigma_{t-1}) \end{cases} \tag{8}$$

with $\kappa_t := (1+\gamma)^{1/\lambda} z_t^{-(1-\lambda)/\lambda}$. A unique steady state point exists: $(\beta^*, z^*, \sigma^*) = (1+\gamma, 1+\gamma, \gamma(2+\gamma)/(1+\gamma)^2)$. The analysis in the next section will show that a local bifurcation occurs implying that some combinations of parameters (γ, λ) allow for stability, while other combinations will lead to a local instability result.

4 Dynamics

The linearization of (8) in the vicinity of point (β^*, z^*, σ^*) yields the following matricial system,

$$\begin{bmatrix} \beta_t - \beta^* \\ z_t - z^* \\ \sigma_t - \sigma^* \end{bmatrix} \simeq \begin{bmatrix} \frac{1}{(1+\gamma)^2} & -\frac{1-\lambda}{\lambda}\frac{\gamma(2+\gamma)}{(1+\gamma)^2} & 0 \\ 1 & 0 & 0 \\ 0 & -\frac{2(1-\lambda)}{\lambda}\frac{\gamma(2+\gamma)}{(1+\gamma)^5} & \frac{1}{(1+\gamma)^2} \end{bmatrix} \cdot \begin{bmatrix} \beta_{t-1} - \beta^* \\ z_{t-1} - z^* \\ \sigma_{t-1} - \sigma^* \end{bmatrix} \tag{9}$$

Applying the center manifold theorem, the stability analysis can focus on the sub-matrix of the matrix in (9) containing the two first rows and the two first columns; let this matrix be J. Stability conditions $1 + Tr(J) + Det(J) > 0$ and $1 - Tr(J) + Det(J) > 0$ are satisfied $\forall \lambda \in (0, 1), \gamma > 0$. The remaining necessary condition for stability, $1 - Det(J) > 0$, holds under $\lambda > \frac{\gamma(2+\gamma)}{3+2\gamma(2+\gamma)}$. A Neimark-Sacker bifurcation occurs at point $\lambda = \frac{\gamma(2+\gamma)}{3+2\gamma(2+\gamma)}$ (for this combination of parameters the eigenvalues of matrix J become a pair of complex conjugate values with modulus equal to 1). Figure 1 presents the areas of local stability and local instability in the space of parameters; a bifurcation curve separates the two areas.[3]

In figure 1, the role of sticky-information becomes clear: if λ is low (high degree of information stickiness), then convergence to the constant growth rate steady state may not occur; the larger the steady state growth rate of x, the more this result is likely to be evidenced.

Regarding global dynamics, one should inquire about the possible generation of endogenous cycles arising immediately after the bifurcation point. In fact, the Neimark-Sacker bifurcation produces a small region of irregular cycles for relatively high values of γ. In figure 2, we take a 100% growth rate ($\gamma = 1$) and $\lambda = 0.4225$ (each time period, 42.25% of the agents update their information concerning economic performance). The displayed attractor corresponds to the set to which the pair (β_{t-1}, β_t) converges in the long-term.[4]

For the chosen values, we can confirm the existence of chaotic motion by computing the corresponding Lyapunov characteristic exponents. A positive Lyapunov exponent indicates the presence of sensitive dependence on initial conditions, a basic property of any chaotic system. In fact, in the case in appreciation the system involves three Lyapunov exponents and the largest one is positive: $\ell_1 = -1.597$; $\ell_2 = -0.032$; $\ell_3 = 0.004$.

5 Discussion

The proposed model intends to provide a framework that combines the behavior, in terms of expectations formation, of relatively well informed individuals (the ones who have updated their information set recently) with the behavior of the agents that have an outdated notion about how the economy truly performs. The first are able to form expectations with a significant component of perfect foresight; the second ones will, alternatively, resort to a learning process trying to predict the long-term growth rate of the considered variable. The analysis of the dynamic system has revealed that a high degree of information stickiness may imply a loss of stability: convergence to the fixed point in which the assumed variable grows at a constant rate no longer holds if learning significantly prevails over the formation of expectations under perfect foresight. Moreover, nonlinear dynamics arise when the Neimark-Sacker bifurcation is crossed; these endogenous fluctuations indicate that for specific levels of information stickiness and learning requirements, the expected value of the variable under consideration will display a bounded instability behavior.

The bifurcation and the endogenous fluctuations outcome arise, in this particular setup, because we have considered a non-stationary variable. As Tuinstra and Wagener (2007) highlight, the ordinary least squares learning algorithm is a decreasing gain algorithm; this means that as one adds progressively more data on the value of the variable, the new observations will have a decreasing or progressively small impact. For a stationary series, this reasoning

[3]Figure 1 is presented at the end of the paper.

[4]Figure 2 is presented at the end of the paper.

means that the gain sequence (σ_t defined in section 3) will asymptotically fall to zero, implying a long-run outcome of perfect foresight. In this case, the learning process is efficient or optimal: as new information is collected, the expectations are improved and convergence to a rational expectations fixed point result is fulfilled. A departure from stability and the possible formation of endogenous cycles in the long-run [the outcome that Bullard (1994) designates as learning equilibria] is possible only if one takes a non-stationary endogenous variable; non-stationarity implies that in the long-run the gain sequence remains constant above zero, i.e., even after many observations, one additional observation is still relevant in the formation of expectations. In this case, learning is not perfect, it involves an everlasting effort on evaluating past observations to adjust expectations as the result of incoming new information. Thus, the analysis in the previous sections allows for evaluating stability of a variable that is supposed to grow at a constant rate in the steady state; otherwise, we would have a trivial process of convergence to the steady state independently of parameter values, i.e., independently of the degree of sticky-information.

The previous argument is better illustrated by applying the regression over the stationary series $y_t := x_{t+1}/x_t$. In this case, the perceived law of motion is $\widehat{y_t} = \beta$. The estimation of β (resorting, again, to OLS) yields now $\beta_{t+1} = \frac{1}{t}\sum_{s=1}^{t} \frac{y_s}{y_{s-1}} = \left(1 - \frac{1}{t}\right) \beta_t + \frac{1}{t} y_{t-1}$. Rearranging the expression one gets: $\beta_{t+1} = \beta_t + \frac{1}{t}\left(y_{t-1} - \beta_t\right)$. This difference equation is similar to the first equation of the learning system in (8); the difference is that the gain sequence assumes now a specific value $\sigma_t = 1/t$. This gain sequence decreases in time and falls asymptotically to zero: $\lim_{t\to\infty} \frac{1}{t} = 0$. Therefore, we confirm that the regression over the stationary variable implies a fully successful learning result: in the long-term there is perfect convergence towards the rational expectations equilibrium.

A last remark involves the issue of forecast errors. Learning procedures are intended to eliminate systematic forecast errors. If these occur, then agents will try to improve their forecasting results by switching to a learning algorithm that performs better or by changing the adopted PLM. However, as argued by Schonhofer (1999), when the outcome of learning is a chaotic time series, forecast errors will be irregular and complex becoming hard to distinguish from pure noise. Deterministic series originating in chaotic processes have some basic properties (e.g. sample average and sample autocorrelation coefficients) that are similar to the ones generated by stochastic processes, and thus agents may be unable to perceive that errors are systematic, interpreting them just as random behavior. In this case, agents cannot learn the perfect foresight long-run outcome and fluctuations persist over time. In the words of Schonhofer (2001), agents cannot learn their way out of chaos. Also Grandmont (1998) calls the attention to this fact: agents incur in self-fulfilling mistakes because they interpret the reality in a simple way (variables would presumably follow simple stochastic processes), when reality is in fact complex (fluctuations are, at least partially, the outcome of complicated deterministic dynamics). Mistakes are self-fulfilling because although systematic errors indeed persist, the regularities that agents are able to extrapolate resorting to simple stochastic rules are similar to the ones of the deterministic process, and thus errors are never perceived as possible to be removed. The literature on consistent expectations equilibria [Sorger (1998), Hommes and Sorger (1998), Hommes and Rosser (2000), among others] makes use of the self-fulfilling mistake concept to analyze the implications of applying simple rules to generate forecasts in a complex world.

Relatively to the model in this short note we can, in the light of the above arguments, claim that the observed fluctuations are likely to persist in time if information is sufficiently sticky to push the model into the chaotic zone. One can illustrate this argument by looking directly at the forecast errors of the specified model, which are $e_t := x_t - \beta_{t-1} x_{t-1}$. Recovering relation (7), we re-write the expression as $\frac{e_t}{x_t} \simeq 1 - \left(\frac{\beta_{t-1}}{1+\gamma}\right)^{1/\lambda}$. Forecast errors will grow over time without bound (this is a direct implication of running a least squares regression on a non-stationary time series); thus, to confirm that under chaos no significant autocorrelations are found in forecast errors, we assume the stationary ratio $\frac{e_t}{x_t}$. Resorting to the numerical example in section 4, we consider a series with $1,000$ consecutive observations on β and compute sample autocorrelations with various lags; the results are presented in the following table:

lag	autocorrelation
1	-0.01211
2	-0.01667
3	-0.01667
4	-0.01648
5	0.33708
6	0.02265
7	-0.01634
8	-0.01665
9	-0.01661
10	-0.00640

The autocorrelations in the table are negative or positive but, in both cases, very low; thus, we conclude that no structural pattern exists in the forecast errors and therefore self-fulfilling mistakes are likely to be found in the agents' behavior.

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Figures

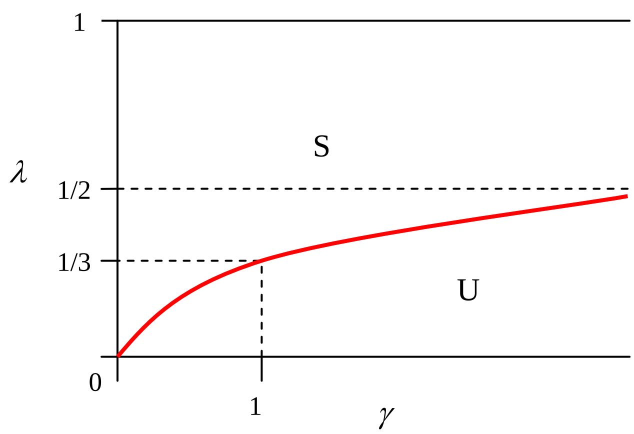

Fig. 1 – Stability in the space of parameters.

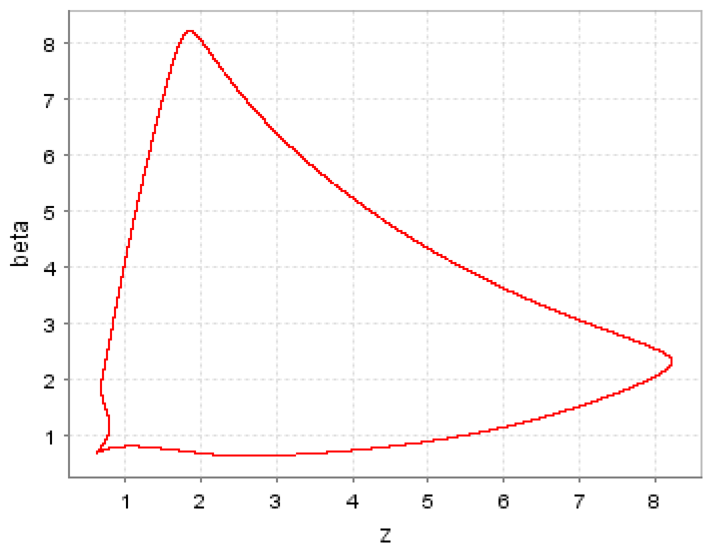

Fig. 2 – Attractor (β_{t-1},β_t); γ=1, λ=0.4225.