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A characterization of the maximin social ordering

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Abstract

This note shows that the maximin social ordering, which is wellknown in the social choice theory, is characterized by Hammond Equity, Continuity, and Weak Pareto Principle.

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1. Introduction

This note provides an axiomatic characterization of *the weighted maximin social ordering*, which compares utility vectors based on the least weighted utilities of the utility vectors. When the weights are symmetric, this ordering becomes the maximin social orderings. We introduce axioms named *α-Hammond Equity, Continuity*, and *Weak Pareto Principle.* We follow the strategy of proof by Fleurbaey (2005, Theorem 3), who gives a characterization of the Pazner-Schmeidler social ordering in the model of exchange economy. Note that the Pazner-Schmeidler social ordering is a kind of maximin social ordering. [1]

As long as we know, there are few studies to characterize the maximin social ordering over utility vectors. Strasnick (1976) characterizes the social ordering by using axioms named *Anonimity, Nonnegativity* and *Unanimity.* [2] Bosmans and Ooghe (2006) prove that the social ordering is axiomatized by *Anonymity*, *Hammond Equity, Continuity*, and *Weak Pareto.* In contrast, our result implies that, when the weights are symmetric, the maximin social ordering is characterized by *Hammond Equity, Continuity*, and *Weak Pareto.* Hence, our characterization does not need *Anonymity.*

The remaining part of this note is as follows. Section 2 gives notation and definitions. Section 3 provides our characterization result.

2. Notation

Let $N = \{1, ..., n\}$ be the set of individuals. $\mathbb{R}$ and $\mathbb{N}$ are, respectively, the sets of real numbers and natural numbers. $X^N = \mathbb{R}^n$ denotes the n-dimensional utlity space.

A social ranking over utility vectors is denoted by R. For any two utility vectors $u, v \in X^N$, $[uRv]$ is interpreted as "u is socially at least as good as v." Symmetric and asymmetric parts of R are denoted by I and P, respectively. A binary relation is a quasi-ordering if it satisfies reflexivity and transitivity. A binary relation is an ordering if it satisfies completeness and transitivity.

We define the α-maximin social ordering.

[1] There are many maximin types of social ordering in economic environments, because of various ways of interpersonal comparison.

[2] He used also an axiom *Neutrality*, though this is known to be redundant.

Definition: Given a vector of weight $\alpha = (\alpha_1, ..., \alpha_n) \in \mathbb{R}^n_{++}$, a social ranking $R_M(\alpha)$ is *the α-maximin social ordering* defined as follows: For any $u, v \in X^N$,

$$uR_M(\alpha)v \Longleftrightarrow \min_{i\in N}\{\alpha_i u_i\} \geq \min_{i\in N}\{\alpha_i v_i\}.$$

This social ordering compares utility vectors, u and v, based on the least weighted utilities, $\min_{i\in N}\{\alpha_i u_i\}$ and $\min_{i\in N}\{\alpha_i v_i\}$. Note that, when $\alpha = (1, ..., 1)$, this soical ordering becomes the *maximin social ordering.*

We introduce the axioms to characterize the maximin social ordering.

Weak Pareto: For all $u, v \in X^N$, if $u_i > v_i$ for all $i \in N$, then uPv.

α-Hammond Equity: Given $\alpha = (\alpha_1, ..., \alpha_n) \in \mathbb{R}^n_{++}$, for all $u, v \in X^N$, if $\alpha_i v_i > \alpha_i u_i > \alpha_j u_j > \alpha_j v_j$ for some $i, j \in N$, and $u_k = v_k$ for all $k \neq i, j$, then uRv.

Continuity: For all $u \in X^N$, if a sequence of vectors $(v^k)_{k\in\mathbb{N}}$ converges to $v \in X^N$ and uRv^k (resp. $v^k Ru$) holds for all $k \in \mathbb{N}$, then uRv (resp. vRu).

Weak Pareto requires that, if all agents are better in one situation u than another v, the former should be socially prefered to the latter.

α-Hammond Equity is a modified version of *Hammond Equity* proposed by Hammond (1976). This axiom insists that a reduction of inequality in *weighted* utilities between two individuals should be socially accepted. Note that, when $\alpha = (1, ..., 1)$, the axiom becomes *Hammond Equity.*

Continuity requires social orderings to be continuous.

3. Characterization

Theorem Suppose that R is a quasi-ordering. Then, R satisfies *α-Hammond Equity, Weak Pareto* and *Continuity* if and only if $R = R_M(\alpha)$.

Proof. It is obvious that α-weighted maximin social ordering satisfies the axioms in the Theorem. We show the converse result. Suppose that a social quasi-ordering R satisfies the axioms. We first prove that, for any ulitity vectors $u, v \in X^N$

$$\min_{i\in N}\{\alpha_i u_i\} > \min_{i\in N}\{\alpha_i v_i\} \Longrightarrow uPv. \tag{1}$$

We first show that one can go from v to u through a sequence of utility vectors $z^1, ..., z^T$ such that $z^1 = v$, $z^T = u$, and for all $t = 1, ..., T-1$, either
(Case 1) $z_i^{t+1} > z_i^t$ for all $i \in N$, or
(Case 2) for two agents i and j,

$$\alpha_i z_i^t > \alpha_i z_i^{t+1} > \alpha_j z_j^{t+1} > \alpha_j z_j^t,$$

and for all other agents k, $z_k^{t+1} > z_k^t$.

We prove this fact.[3] Let m be an agent such that $\alpha_m v_m = \min_i\{\alpha_i v_i\}$. Define $S = \{i | \alpha_i v_i > \alpha_m v_m\}$ and $M = \min_{i \in S}\{\alpha_i v_i\}$. Let $\epsilon > 0$ be such that

$$\epsilon < \frac{1}{n}\Big(\min\Big\{M, \min_{i\in N}\{\alpha_i u_i\}\Big\} - \alpha_m v_m\Big).$$

Let $T = |S| + 2$ and s be a bijection from $\{1, ..., |S|\}$ to S. At every step $t = 1, ..., T-2$, let
(a) $\alpha_i z_i^{t+1} = \alpha_m v_m + (t+1)\epsilon$ for $i = s(t) \in S$,
(b) $\alpha_k z_k^{t+1} = z_k^t + \epsilon$ for all $k \neq i$. (In particular, $\alpha_m z_m^{t+1} = \alpha_m v_m + t\epsilon$.)

For $t = 1, ..., T-2$, the step from z^t to z^{t+1} corresponds to (Case 2) with $i = s(t)$ and $j = m$, since

$$\alpha_m z_m^t < \alpha_m z_m^{t+1} = \alpha_m v_m + t\epsilon < z_i^{t+1} = \alpha_m v_m + (t+1)\epsilon < z_i^t, \qquad (2)$$

where the last inequality is derived from

$$\alpha_m v_m + (t+1)\epsilon < \alpha_m v_m + \frac{t+1}{n}(M - \alpha_m v_m) \le M \le \alpha_i z_i^t.$$

The last step from z^{T-1} to $z^T = u$ corresponds to (Case 1). This is because, for all i,

$$\alpha_i z_i^{T-1} \le \alpha_m v_m + (T-1)\epsilon < \alpha_m v_m + n\epsilon < \min_i\{\alpha_i u_i\},$$

where the last inequality follows the assumption regarding ϵ above.

Now we prove (1). For all steps $t = 1, ..., T-2$, let ϵ' be such that

$$\alpha_i z_i^{t+1} - \epsilon' > \alpha_m z_m^{t+1} - \epsilon' > \alpha_m z_m^t,$$

where $i = s(t)$. By (2) and *α-Hammond Equity*,

$$(z_i^{t+1} - \epsilon', z_m^{t+1} - \epsilon', z_{-im}^t) R z^t.$$

[3]The proof is essentially due to Fleurbaey (2005, proof of Theorem 3, Step 1).

By *Weak Pareto*,

$$z^{t+1} P (z_i^{t+1} - \epsilon', z_m^{t+1} - \epsilon', z_{-im}^t).$$

By transitivity, $z^{t+1} P z^t$. Moreover, by *Weak Pareto*, $z^T = uPz^{T-1}$. By transitivity, uPv. Thus, (1) has been shown.

From (1) and the usual argument of *Continuity*, we can easily show that, for any $u, v \in X^N$,

$$\min_{i \in N} \alpha_i u_i = \min_{i \in N} \alpha_i v_i \Longrightarrow uIv. \tag{3}$$

By (1) and (3), we have completed the proof. □

The axioms in the Theorem are clearly independent.

References

[1] Bosmans, K., and Ooghe, E., (2006), "A Characterizaion of Maximin," Mimeo, Katholieke Universiteit Leuven.

[2] Fleurbaey, M., (2005), "The Pazner-Schmeidler Social Ordering: A Defense," *Review of Economic Design*, 9, pp. 145-166.

[3] Hammond, P., J., (1976), "Equity, Arrow 's Conditions, and Rawls ' difference principle," *Econometrica* 44, pp. 793-803.

[4] Strasnick, S., (1976), "Social Choice and the Derivation of Rawls's Difference Principle," *Journal of Philosophy,* 73-4, pp. 85-99.