

Volume 30, Issue 3

Anatomy of production functions: a technological menu and a choice of the best technology

Vladimir Matveenko

Institute for Economics and Mathematics - Russian Academy of Sciences

Abstract

Jones (2005) proposed microfoundations for the Cobb-Douglas production function. We show that Jones' technological menu is a special case of a concept of support set discussed by Matveenko (1997) and Rubinov, Glover (1998) by use of a duality approach. We use this approach to clarify the relation between different production functions and technological menus. Also we construct an "ideas model" generating CES production function.

The author is grateful for useful comments from an anonymous referee, Jakub Growiec, Charles Jones, Georgii Kleiner, Alexander Rubinov, and Ping Wang.

Citation: Vladimir Matveenko, (2010) "Anatomy of production functions: a technological menu and a choice of the best technology", *Economics Bulletin*, Vol. 30 no.3 pp. 1906-1913.

Submitted: Mar 23 2010. **Published:** July 20, 2010.

1. Introduction

An important topic in macroeconomics and growth theory concerns microfoundations of basic classes of production functions. A number of authors argue that production function (which has received an attribute ‘global’ in recent publications) is not a primary economic object but a result of an optimal choice of a ‘local’ technology from a given technological menu (Matveenko 1997, Rubinov and Glover 1998, Jones 2005, Caselli and Coleman 2006, Growiec 2008). This approach perfectly matches with a view that, given a combination of production factors, only one local technology can be used efficiently¹.

In particular, Matveenko (1997) and Rubinov and Glover (1998), by use of a duality approach, showed that each global n -factor production function, F , with constant returns to scale (CRS) can be represented as an optimal choice of a local Leontief technology from a menu (a set of technologies) corresponding to the function F .

Later Jones (2005) indicated a similar representation of a 2-factor CRS global production function:

$$F(K, L, N) = \max_{a, b: H(a, b) = N} \tilde{F}(bK, aL).$$

Here \tilde{F} is a local production function with an elasticity of substitution less than one, N is a parameter characterizing available technologies, and the set

$$\{(a, b) : H(a, b) = N\}$$

is a technological menu where technological parameters are chosen from. Under the following technological menu:

$$H(a, b) = a^\alpha b^\beta = N, \quad (1)$$

where $\alpha > 0, \beta > 0$, Jones has received the global Cobb-Douglas function:

$$F(K, L, N) = N^{\frac{1}{\alpha+\beta}} K^{\frac{\beta}{\alpha+\beta}} L^{\frac{\alpha}{\alpha+\beta}}.$$

The present paper develops this approach in several directions. We prove that for each n -factor neoclassical global production function F there exists a unique technological menu consisting of Leontief local technologies and generating F . Basic properties of technological menus are studied. A simple method for indicating technological menus is proposed. As examples, technological menus for the Cobb-Douglas and the CES global production functions are constructed. A case of local CES functions is also considered.

One more result of the paper concerns Jones (2005) “ideas model” based on the Pareto probability distribution and considered as a microfoundation for the global Cobb-Douglas function; its modification was recently constructed by Growiec (2008). We propose a simpler modification leading to the CES global production function. Different microfoundations also leading to the CES function have been proposed by Acemoglu (2003).

2. Technological menus and their properties

Let $i = 1, \dots, n$ be factors of production. We will consider a family of local production functions $\varphi(l, x)$; each of them is characterized by fixed technological coefficients (factor efficiencies) $l_i, i = 1, \dots, n$. A basic case is the Leontief local production function

¹ This view is distinctly formulated by (Basu and Weil 1998) who argue that “each technology is... appropriate for one and only one capital-labor ratio”. This idea is close to a concept of localized technological change (Atkinson and Stiglitz 1969, Nelson and Winter 1982, ch. 9, Stiglitz 1989, Antonelli 1995, 2008).

$\varphi(l, x) = \min_{i=1, \dots, n} l_i x_i$. Let $F(x)$ be a global neoclassical production function². A set $\Lambda = \{l = (l_1, \dots, l_n)\}$ is called a technological menu generating the global production function $F(x)$ if

$$F(x) = \max_{l \in \Lambda} \varphi(l, x). \quad (2)$$

An economic meaning of this notion is quite transparent. A firm (or a country) has available a set of local technologies Λ . Given a vector of production factors $x = (x_1, \dots, x_n)$ it chooses a technology l from Λ to achieve the maximum³ output $F(x)$. In result the global production function $F(x)$ is formed by use of the family of local production functions.

A number of natural questions arise. Is each global production function generated by a technological menu? Is the technological menu, generating a concrete global production function, unique? If yes, what is the structure of the menu? The following Theorem 1 provides an exhaustive answer to these questions for the case of local Leontief technologies.

Let M_1 be a unit level surface of the function $F(x)$:

$$M_1 = \{x : F(x) = 1\},$$

i.e. the set of all vectors of production factors which provide a unit output.

We will narrow the domain of production functions in some way. We will consider production functions defined on the space R_{++}^n which consists of positive n -dimensional vectors and the origin⁴.

This narrowing allows us to consider for each vector of factors, $x \in M_1$, a vector of inverse elements:

$$x^- = (x_1^{-1}, x_2^{-1}, \dots, x_n^{-1}).$$

Its economic meaning is that x_i^{-1} is an average product of the i -th factor. (Evidently, $x_i^{-1} = F(x)/x_i$ as soon as $F(x) = 1$ for $x \in M_1$).

We will see that the set

$$\Lambda_1 = \{l : l = x^-, x \in M_1\} \quad (3)$$

(known as a support set – see (Matveenko, 1997, Rubinov and Glover, 1998)) is a unique technological menu generating the global production function $F(x)$ under Leontief local technologies.

There is an equivalent way to describe the technological menu. For the global production function $F(x)$ let us define an auxiliary function

$$F^\circ(l) = \frac{1}{F\left(\frac{1}{l_1}, \dots, \frac{1}{l_n}\right)}, \quad (4)$$

so called conjugate function. An advantage of its use is that it is easily computable (see examples in Section 3). The technological menu (3) can be found as a unit level surface of the conjugate function:

$$\Lambda_1 = \{l : F^\circ(l) = 1\}.$$

Both the technological menu and the conjugate functions have simple economic interpretations. It is easy to verify that the technological menu Λ_1 generating the global

² The functions are assumed to be non-negative, continuous, increasing and possessing CRS.

³ This corresponds in full to the concept of (Basu and Weil 1998) – see footnote 1.

⁴ Thereby, we will not consider points where at least one factor is not used. This does not contract the class of production functions.

production function F consists of all vectors l with coordinates equal to average products of factors that are possible given F . For example, for a 2-factor global production function $F(K, L)$ the technological menu Λ_1 consists of all admissible pairs $(F(K, L)/K, F(K, L)/L)$ of average capital and labor productivities.

An economic interpretation of the conjugate function $F^\circ(l)$ is the following. For each $l = (l_1, \dots, l_n)$ it shows a minimum value of a total factor productivity (TFP) A such that the function $AF(\cdot)$ makes admissible average products l_1, \dots, l_n .

THEOREM 1. *The set Λ_1 is a unique technological menu generating the global production function F .*

Proves of theorems are provided in Section 6.

One more question: is it possible, knowing a form of a technological menu, to predetermine properties of the global production function generated by this menu? A partial answer is given by the following Theorem 2, where, for the sake of simplicity, only a 2-dimensional case is considered.

Let us define a set of all available technologies, $\tilde{\Lambda}$, which includes the technological menu Λ_1 as well as all worse technologies:

$$\tilde{\Lambda} = \{l : l_K > 0, l_L > 0, F^\circ(l) \leq 1\}$$

THEOREM 2. *If the set $\tilde{\Lambda}$ of available technologies is convex then the elasticity of substitution σ of the global production function $F(K, L)$ in any point $x = (K, L)$ is less than $1/2$.*

Notice that here the production function can possess different elasticities of substitution in different points, nevertheless they all have to be less than $1/2$.

On an intuitive level the link between a form of the set of available technologies and a size of the elasticity of substitution of the global production function can be explained as follows. A low elasticity of substitution means a limited possibility to change technologies. A convexity of the set $\tilde{\Lambda}$ just restricts a possibility of changing technologies: a technology $l \in \Lambda_1$ may be changed for a technology $\tilde{l} \in \Lambda_1$ if and only if there exists a chord connecting l and \tilde{l} and situated in interior of the set $\tilde{\Lambda}$.

3. Examples

For the global 2-factor Cobb-Douglas production function,

$$F(K, L, \bar{N}) = \bar{N} K^{\bar{\beta}} L^{\bar{\alpha}} \quad (\text{where } \bar{\alpha} + \bar{\beta} = 1, 0 < \bar{\alpha} < 1),$$

its conjugate function is

$$F^\circ(l_K, l_L) = \frac{1}{\bar{N} l_K^{-\bar{\beta}} l_L^{-\bar{\alpha}}},$$

hence the technological menu is

$$\Lambda_1 = \left\{ l : \bar{N} l_K^{-\bar{\beta}} l_L^{-\bar{\alpha}} = 1 \right\} = \left\{ l : l_K^{\bar{\beta}} l_L^{\bar{\alpha}} = \bar{N} \right\};$$

this coincides with Jones' menu (1).

For the global 2-factor CES production function,

$$F(K, L, N) = N \left[A L^{-r} + B K^{-r} \right]^{1/r}$$

with $A, B > 0, -1 \leq r, r \neq 0$, the conjugate function is:

$$F^\circ(l_K, l_L) = N^{-1} \left[A l_L^r + B l_K^r \right]^{1/r},$$

and the technological menu is:

$$\Lambda_1 = \{l : (Al_L^r + Bl_K^r)^{1/r} = N\}.$$

4. Local CES function

Similar results concerning the technological menus are also true in case of local CES function:

$$\varphi(l, x) = (l_1 x_1^p + \dots + l_n x_n^p)^{\frac{1}{p}},$$

where $p < 0$ is a fixed parameter⁵.

THEOREM 3. *With local CES function,*

(i) *A global production function generated by the technological menu*

$$\Lambda = \left\{ \sum_{i=1}^n l_i^\alpha a_i^{1-\alpha} = 1 \right\} \quad (\alpha < 1),$$

has a CES form.

(ii) *A global production function generated by the technological menu*

$$\Lambda = \left\{ l : l_1^{\theta_1} \dots l_n^{\theta_n} = B \right\} \quad (\text{where } B > 0, 0 < \theta_i < 1, i = 1, \dots, n, \sum_{i=1}^n \theta_i = 1)$$

has a Cobb-Douglas form.

5. Technological ideas model

Jones (2005), looking for microfoundations of global production functions, proposed a model of technological ideas. An idea i means the use of Leontief technological coefficients a_i, b_i which are random and independent; precisely, they are described by independent Pareto distributions:

$$P\{a_i \leq a\} = 1 - \left(\frac{a}{\gamma_a}\right)^{-\alpha}, \quad P\{b_i \leq b\} = 1 - \left(\frac{b}{\gamma_b}\right)^{-\beta},$$

where $a \geq \gamma_a > 0, b \geq \gamma_b > 0, \alpha > 0, \beta > 0, \alpha + \beta > 1$, and their joint distribution is:

$$G_1(b, a) = P\{b_i > b, a_i > a\} = \left(\frac{b}{\gamma_b}\right)^{-\beta} \left(\frac{a}{\gamma_a}\right)^{-\alpha}.$$

However, the independency assumption is not motivated at all. Let us make an alternative assumption: an idea is a pair of interdependent technological coefficients a_i, b_i . The following joint probability distribution can be used as a simple model:

$$G_2(b, a) = P\{b_i > b, a_i > a\} = \left[\lambda \left(\frac{b}{\gamma_b}\right)^{-h} + (1 - \lambda) \left(\frac{a}{\gamma_a}\right)^{-h} \right]^s,$$

where $a \geq \gamma_a > 0, b \geq \gamma_b > 0, 0 < \lambda < 1, h < 0$ or $0 < h < 1, hs > 1$.⁶

⁵ The condition on p makes possible the part (ii) of Theorem 3.

⁶ The conditions on h and s are imposed to provide appropriate properties of a production function and a probability distribution below.

The functions G_1, G_2 have resembling properties, moreover, under $h \rightarrow 0$ and $hs = \text{const}$, a conversion $G_2(b, a) \rightarrow G_1(b, a)$ takes place, where parameters of the functions G_1, G_2 are linked by relations $\beta = \lambda hs, \alpha = (1 - \lambda)hs$.

Assuming the distribution G_2 ,

$$P\{Y_i > \tilde{y}\} = P\{b_i K > \tilde{y}, a_i L > \tilde{y}\} = \tilde{y}^{-hs} [\lambda (K\gamma_b)^h + (1 - \lambda)(L\gamma_a)^h]^s.$$

With N ideas,

$$P\{Y \leq \tilde{y}\} = \left[1 - \tilde{y}^{-hs} [\lambda (K\gamma_b)^h + (1 - \lambda)(L\gamma_a)^h]^s\right]^N.$$

By using a normalization,

$$z_N = \left[\lambda (K\gamma_b)^h + (1 - \lambda)(L\gamma_a)^h\right]^{\frac{1}{h}} N^{\frac{1}{hs}},$$

it is easy to receive:

$$P\{Y \leq z_N \tilde{y}\} = \left(1 - \frac{\tilde{y}^{-hs}}{N}\right)^N \rightarrow \exp(-\tilde{y}^{-hs}).$$

As well as in the Jones' case, with large N ,

$$Y \approx z_N \varepsilon,$$

where ε is a random variable described by the Frechet distribution.

Thus, when the number of ideas is great, we come to a CES production function.

6. Proves of theorems

For n -dimensional vectors, $x \geq y$ means that $x_i \geq y_i$; $x > y$ means that $x_i > y_i$ ($i = 1, \dots, n$). A function f is called increasing if $x > y$ implies $f(x) > f(y)$.

As a preliminary we prove the following Lemma.

LEMMA. *If $F(x)$ is an increasing function homogeneous of the power α then*

$$\varphi(l, x) \leq F(x) = 1$$

for each $x \in M_1, l \in \Lambda_1$.

Proof. Let $x \in M_1$. Let us prove that $\varphi(l, x) \leq 1$ for any $l \in \Lambda_1$. Assume the opposite: $\varphi(l, x) > 1$ for some $l \in \Lambda_1$. Then

$$l_i x_i > 1, i = 1, \dots, n,$$

and hence $x > l^-$. A number $\lambda > 1$ can be picked up such that $x > \lambda l^-$. Then

$$F(x) > F(\lambda l^-) = \lambda^\alpha F(l^-) > F(l^-),$$

which contradicts to the belonging $x \in M_1, l^- \in M_1$.

Q.E.D

Proof of Theorem 1. Each vector $x \in R_{++}^n$ can be represented in the form $x = F(x)\bar{x}$ where $\bar{x} \in M_1$. For any $l \in \Lambda_1$ it follows from Lemma that

$$\varphi(l, \bar{x}) \leq F(\bar{x}) = 1 = \varphi(\bar{x}^-, \bar{x}).$$

We use the Lemma again to receive

$$\varphi(l, x) = F(x)\varphi(l, \bar{x}) \leq F(x), l \in \Lambda_1;$$

$$\varphi(\bar{x}^-, x) = F(x)\varphi(\bar{x}^-, \bar{x}) = F(x).$$

This means the validity of (2).

To prove the uniqueness, let $\bar{\Lambda}$ be another technological menu generating the same global production function $F(x)$. Let us consider in $\bar{\Lambda}$ points $\arg \max_{l \in \bar{\Lambda}} \varphi(l, x)$ for all $x > 0$.

No one of these points can lie below Λ_1 (i.e. inside $\tilde{\Lambda}$). On the other hand, if any of these points \bar{l} lies above Λ , then there exists a point $l \in \Lambda$ such that $l < \bar{l}$. For the point $x = l^- = (l_1^{-1}, \dots, l_n^{-1})$,

$$F(x) = \varphi(l, x) < \varphi(\bar{l}, x) = F(x).$$

This contradiction implies $\bar{\Lambda} = \Lambda$.

Q.E.D.

Proof of Theorem 2. Convexity of $\tilde{\Lambda}$ is equivalent to concavity of the function $l_L(l_K)$ and, hence, to the inequality:

$$\frac{d^2 l_L}{dl_K^2} < 0.$$

By use of representation $F(K, L) = Lf(k)$, $l_L = f(k)$, $l_K = f(k)/k$ where $k = K/L$, we find

$$\frac{d^2 l_L}{dl_K^2} = \frac{2k(f'(k))^2 - kf(k)f''(k) - 2f(k)f'(k)}{(f'(k)k - f(k))^3} k^3.$$

The denominator is negative, so the sign of the fraction coincides with the sign of the expression

$$\Phi = 2f'(k)(f(k) - kf'(k)) + kf(k)f''(k)$$

Recalling the well-known formula for the elasticity of substitution of production function $F(K, L)$,

$$\sigma = \frac{f'(k)(f(k) - kf'(k))}{-kf(k)f''(k)},$$

we can rewrite Φ as

$$\Phi = -kf(k)f''(k)(2\sigma - 1).$$

As soon as $kf(k)f''(k) < 0$ for neoclassical production functions, the sign of Φ coincides with the sign of $(2\sigma - 1)$, so the convexity of $\tilde{\Lambda}$ implies $2\sigma - 1 < 0$.

Q.E.D.

Proof of Theorem 3. (i) First let us prove the validity of the equation (2) for any $x \in M_1$. The Lagrange multipliers method provides the first order conditions:

$$a_1^{\frac{1}{q-p}} l_1^{\frac{1}{q-p}} x_1 = \dots = a_n^{\frac{1}{q-p}} l_n^{\frac{1}{q-p}} x_n, \quad (5)$$

$$a_1^{-\frac{p}{q-p}} l_1^{\frac{q}{q-p}} + \dots + a_n^{-\frac{p}{q-p}} l_n^{\frac{q}{q-p}} = 1. \quad (6)$$

In (5) denote $a_i^{\frac{1}{q-p}} l_i^{\frac{1}{q-p}} x_i = c$ ($i = 1, \dots, n$). It follows from (6) that $c = 1$. Hence, in the maximum point:

$$l_i = a_i x_i^{q-p}, \quad i = 1, \dots, n,$$

and (2) is true.

Each vector $x > 0$ can be represented as $x = \mu \bar{x}$, where $\bar{x} \in M_1$. Because of the homogeneity,

$$F(x) = F(\mu \bar{x}) = \mu F(\bar{x}) = \mu \max_{l \in \Lambda} \varphi(l, \bar{x}) = \max_{l \in \Lambda} \varphi(l, x).$$

(ii) is proved in a similar way.
Q.E.D.

References

- Acemoglu, D. (2003) "Labor- and capital-augmenting technical change" *Journal of the European Economic Association* **1**, 1-37.
- Antonelli, C. (1995) *The Economics of Localized Technological Change and Industrial Dynamics*, Kluwer: Dordrecht.
- Antonelli, C. (2008) *Localized technological change: Towards the economics of complexity*, Routledge: Abingdon.
- Atkinson, A. and J.E. Stiglitz (1969) "A new view of technological change" *Economic Journal* **79**, 46-49.
- Basu, D. and D.N. Weil (1998) "Appropriate technology and growth" *Quarterly Journal of Economics* **113**, 1025-54.
- Caselli, F. and W.T. Coleman (2006) "The World technology frontier" *American Economic Review* **96**, 499-522.
- Growiec, J. (2008) "A new class of production functions and an argument against purely labor-augmenting technical change" *International Journal of Economic Theory* **4**, 483-502.
- Jones, C.I. (2005) "The shape of production function and the direction of technical change" *Quarterly Journal of Economics* **120**, 517-49.
- Matveenko, V (1997) "On a dual representation of CRS functions by use of Leontief functions" in *Proceedings of the First International Conference on Mathematical Economics, Non-Smooth Analysis, and Informatics*, Institute of Mathematics and Mechanics Azerbaijan National Academy of Sciences: Baku, 160-165.
- Nelson, R. and S.G. Winter (1982) *An Evolutionary Theory of Economic Change*, Delknap Press/Harvard University Press: Cambridge.
- Rubinov, A.M. and B.M. Glover (1998) "Duality for increasing positively homogeneous functions and normal sets" *Operations Research* **12**, 105-23.
- Stiglitz, J.E. (1989) "Markets, market failures, and development" *American Economic Review* **79**, Papers and Proceedings, 197-203.