Economics Bulletin

Volume 30, Issue 3

A note on Kalman filter approach to solution of rational expectations models

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Abstract

In this note, a class of nonlinear dynamic models under rational expectations is studied. A particular solution is found using a model reference adaptive technique via an extended Kalman filtering algorithm, for which initial conditions knowledge only is required.

I wish to thank Sergiy Zhuk and conference participants in Bielefeld and Salerno for valuable comments and discussions on earlier drafts of this paper. All remaining errors are my own.
Citation: Marco Maria Sorge, (2010) "A note on Kalman filter approach to solution of rational expectations models", *Economics Bulletin*, Vol. 30 no.3 pp. 2002-2009.
Submitted: Mar 08 2010. **Published:** July 30, 2010.

1. Introduction

Since the early work of Muth (1961) and Lucas (1972), the concept of rational expectations (RE) has become the standard tool of modeling expectations in dynamic macroeconomics. It essentially reduces to the assumption that agents collect and make optimal use of all available (pertinent) information as to the economic environment when formulating their forecasts of economic variables of interest (e.g., prices, interest rates, government policies). Since rational expectations need to be model-consistent and are endogenously determined, equilibria of economic systems described by dynamic forward-looking equations are typically non-unique (e.g., Sargent and Wallace, 1973; Taylor, 1977; Blanchard and Kahn, 1980; Broze and Szafarz, 1991; Sims, 2002).

In order to address this issue, Carravetta and Sorge (2010) fully characterize the class of linear time-varying RE models - namely, those displaying no backward-looking dimension - for which a solution can be obtained via a *causal* model forced by a well-identified control function, estimated via a Kalman filter technique. More specifically, the optimal minimum variance estimate of the future state is recursively computed by applying the Kalman filter, fed by the noisy measurements of the state vector, on the autoregressive equation describing the perfect-foresight dynamics of the economy. An exact solution of the actual RE problem is thus determined by using such estimator as control function in a causal (controllable) state system, for which initial conditions knowledge only is required.

The use of dynamic stochastic nonlinear models has increasingly widened in the economic literature over the last years. Since closed-form solutions for such models are rarely available, solution methods typically resort to either simple graphical analysis or elaborate numerical procedures. The introduction of the RE hypothesis for forward-looking models, under which subjective beliefs of decision makers are replaced with the mathematical conditional expectation of some future model's equilibrium state, has made this issue even more difficult to deal with (e.g., Fair and Taylor, 1983; Christiano, 1990; den Haan and Marcet, 1990; Taylor and Uhlig, 1990). This note extends the method developed in Carravetta and Sorge (2010) to solution of a particular class of dynamic forward-looking models in state-space form which are nonlinear in the RE term, using an extended Kalman filtering approach (Anderson and Moore, 1986; Haykin, 1996). To this end, it is organized as follows. In Section 2 the problem we deal with is formally stated, whereas Section 3 presents the estimation algorithm. Section 4 concludes.

2. The model

A general RE model may be characterized by the system of f equations:

$$F(x_-, x, \mathbf{E}(\theta(x_+)|\Omega), v) = 0$$

$$F : \Re^n \times \Re^n \times \Re^q \times \Re^m \to \Re^f, \quad \theta : \Re^n \to \Re^q$$

where $x \in \Re^n$ is a vector of (endogenous) state variables, defined on an appropriate filtered probability space, the initial state $\bar{x}$ being zero-mean Gaussian with covariance matrix P_0. The vector $v \in \Re^m$ collects zero-mean white Gaussian structural (exogenous) shocks with covariance V. Dependence of the system on first lags of the states is summarized in x_-, whereas $\mathbf{E}(\cdot|\cdot)$ denotes conditional (rational) expectations of (some function θ of) future states x_+, based on the information set Ω available to economic agents at the time the forecast is generated.

For given (time-invariant) model structure, the functions F and θ are known. The form of nonlinearities considered in this note accounts for a first-difference temporary equilibrium map in which rational expectations - based on past information - are a nontrivial function of the current states and the fundamental (exogenous) shocks:

$$\mathbf{E}(x_{t+1}|Y_{t-1}) = (l \circ h)(x_t, v_t), \quad \forall t$$

$$l \circ h : \Re^n \times \Re^n \to \Re^n$$

where it is assumed for simplicity $n = m$. For the purpose of the paper, we also make the following:

Assumption 1. *Let h be linear and $l \in C^1$ on the open domain Ξ, with $|D(l(\xi))| \neq 0$ for all $\xi \in \Xi$.*

The requirement that the fundamental shocks be linearly separable from the nonlinear structure of the economy is related to the nature of our solution procedure, which builds upon an adaptive technique - well-known in the literature on stochastic control theory (e.g., Landau, 1979; Kendrick, 1981) - whose reference model is chosen as the autoregressive equation governing the perfect-foresight dynamics of the system economy. The second part of Assumption 1 further restricts the form of nonlinearities admitted by our estimation algorithm to invertible l functions, due to a few technical difficulties, which will be assessed in the next Section.

Accordingly, we will study the vector nonlinear forward-looking difference system under RE of the form[1]:

$$x_{t+1} = f(\mathbf{E}(x_{t+2}|Y_t)) + v_{t+1}, \quad x_0 = \bar{x} \tag{1}$$

where $f := l^{-1}$. The filtration Y_t is generated by the output process $\{y_j, j \leq t\}$ according to:

$$y_t = Cx_t + w_t, \quad y_0 = \bar{y} \tag{2}$$

with $y \in \Re^n$ and w random vector drawn from $\mathbf{N}(0, W)$, which accounts for the output measurement noise, and C a real square matrix. The error sequences $\{v_t, w_t\}$ are assumed to be mutually independent as well as independent of the initial state $\bar{x}$[2].

The RE model (1)-(2) typically admits non-unique solutions for given initial conditions. In the linear case, Carravetta and Sorge (2010) address the multiplicity issue by replacing the unobservable expectational component with a computable one of the required structure, namely the optimal prediction of the perfect-foresight future state whose dynamics are given by (for non-singular state transition matrix B)[3]:

$$x^*_{t+1} = Bx^*_{t+2} + v_{t+1}, \quad x^*_0 = x_0 = \bar{x}, x^*_{-1} = 0 \tag{3}$$

to which the following output equation is attached:

$$y^*_t = Cx^*_t + w_t \tag{4}$$

More specifically, the following is shown (for the time-varying case):

[1] With no loss of generality, we set $h = [I \quad -I]$, with I denoting the $n \times n$ identity matrix.

[2] The parameters of the nonlinear dynamic system, namely f, V, W, and P_0, are taken to be known. Whereas the outputs are observed, the state and error variables are hidden.

[3] This assumption is made for the descriptor system (3) to admit an autoregressive representation.

Lemma 1 (Carravetta and Sorge, 2010). *The linear stochastic forward-looking RE model with noisy observations and time-varying parameters:*

$$x_{t+1} = B_t \boldsymbol{E}(x_{t+2}|Y_t) + v_{t+1}, \quad x_0 = \bar{x}$$

$$y_t = C_t x_t + w_t, \quad y_0 = \bar{y}$$

always admits a solution equal to the one of the causal dynamic stochastic model:

$$x_{t+1} = B_t u_t + v_{t+1}, \quad x_0 = \bar{x}$$

$$y_t = C_t x_t + w_t, \quad y_0 = \bar{y}$$

where the control sequence u_t is set to the optimal minimum variance (prediction) estimate of the perfect-foresight state $\hat{x}^*_{t+2|t}$, fed by the actual measurements $\{y_j, j \leq t\}$ as in (2)[4].

Consider now the causal (memoryless) dynamic stochastic nonlinear model with noisy observations:

$$x_{t+1} = f(u_t) + v_{t+1}, \quad x_0 = \bar{x} \tag{5}$$

$$y_t = Cx_t + w_t \quad y_0 = \bar{y} \tag{6}$$

and let $u' \equiv f(u) \equiv \{f(u_t)\}$ be an admissible - that is, Y_t-adapted - control sequence. From equation (5), the state motion $x = (x_t)$ generated by the input sequence $u' = \mathbf{E}(x^*_{t+2}|Y_t)$ is such that, for any t, the optimal prediction of any future perfect-foresight state, given the measurement $(y_0 \dots y_t)$, is equal to that relative to the actual state. From Assumption 1 and Lemma 1, it readily follows that there exists a RE equilibrium $x = (x_t)$ - that is, a time-dated sequence of (functions of) states and observables in Y_t fulfilling the *non-causal* system (1)-(2) - which is computable via a *causal* model of the form (5)-(6) forced by the optimal prediction estimate of the two-step ahead perfect-foresight state variables:

$$x_{t+1} = f(\mathbf{E}(x^*_{t+2}|Y_t)) + v_{t+1}$$

Consequently, the main issue lies in developing a suitable (locally optimal) state estimation algorithm for the nonlinear dynamic equation:

$$x^*_{t+1} = f(x^*_{t+2}) + v_{t+1} \tag{7}$$

3. The estimation algorithm

Equation (7) represents a non-causal first-difference descriptor model, whose estimation cannot be in general performed by means of simple recursive algorithms. However, owing to the invertibility of f, we may rewrite it as:

$$x^*_{t+2} = \Phi(x^*_{t+1}, v_{t+1}) \tag{8}$$

[4]This was conjectured in De Santis *et al.* (1993). Nonetheless, as demonstrated in Carravetta and Sorge (2010), such conjecture fails to obtain as a general property of dynamic RE models. Moreover, the equilibrium state motion is shown to be the closest, in mean square, to the evolution of the perfect-foresight model (3).

where $\Phi = (l \circ h)(x^*, v)$. This enables us to address the nonlinear filtering problem by means of an extended Kalman filter, fed by the measurements y_t^*[5]. The state equation (8) is thus linearized at $\hat{x}_{t+1}^*$ to yield[6]:

$$x_{t+2}^* \approx \Phi(\hat{x}_{t+1}^*, 0) + D_{x^*}(\Phi)(\hat{x}_{t+1}^*, 0)[x_{t+1}^* - \hat{x}_{t+1}^*] + D_v(\Phi)(\hat{x}_{t+1}^*, 0)v_{t+1} \tag{9}$$

By defining the vector $z_t := [x_t^{*T} \quad x_{t+1}^{*T}]^T$, the linearized state process (9) together with the linear measurement function (4) can be written in vectorial form as:

$$z_{t+1} = Az_t + u_t + Bv_{t+1}, \quad z_0 = [\bar{x}^T \quad l(\bar{x})^T]^T \tag{10}$$

$$y_t^* = \bar{C}z_t + w_t \tag{11}$$

where:

$$A = \begin{pmatrix} 0 & I \\ 0 & D_{x^*}(\Phi)(\hat{x}_{t+1}^*, 0) \end{pmatrix}; \qquad B = \begin{pmatrix} 0 \\ D_v(\Phi)(\hat{x}_{t+1}^*, 0) \end{pmatrix};$$

$$u_t = \begin{pmatrix} 0 & 0 \\ I & -D_{x^*}(\Phi)(\hat{x}_{t+1}^*, 0) \end{pmatrix} \begin{pmatrix} \Phi(\hat{x}_{t+1}^*, 0) \\ \hat{x}_{t+1}^* \end{pmatrix}; \qquad \bar{C} = \begin{pmatrix} C \\ 0 \end{pmatrix}$$

where u_t is regarded to as a Y_{t+1}^*-measurable stochastic input to the augmented state system (10)-(11), which can be dealt with as a deterministic one in order to apply the Kalman filter formula[7]. The optimal filtering estimate is thus obtained as:

$$\hat{z}_{t+1} = \hat{z}_{t+1|t} + K\left(y_{t+1} - \bar{C}\hat{z}_{t+1|t}\right) \tag{12}$$

$$\hat{P}_{t+1}^z = \hat{P}_{t+1|t}^z - K\Sigma K^T \tag{13}$$

where the innovation covariance and the Kalman gain are given respectively as:

$$\Sigma = \bar{C}\hat{P}_{t+1|t}^z\bar{C}^T + W \tag{14}$$

$$K = \hat{P}_{t+1|t}^z\bar{C}^T\Sigma^{-1} \tag{15}$$

The filtering and one-step prediction error covariances are $\hat{P}_t^z = \mathbf{E}\left((z_t - \hat{z}_t)(z_t - \hat{z}_t)^T\right)$ and $\hat{P}_{t|t-1}^z = \mathbf{E}\left((z_t - \hat{z}_{t|t-1})(z_t - \hat{z}_{t|t-1})^T\right)$ respectively, with $\hat{P}_{t+1|t}^z = A\hat{P}_t^z A^T + V$. The initial conditions for the augmented state z_t are readily used to initialize the prediction error covariance $\hat{P}_{0|-1}^z$.

Taking expectations conditional on Y_t^* for equation (10) yields the one-step prediction estimate:

$$\hat{z}_{t+1|t} = A\hat{z}_t + u_t$$

and thus the optimal two-step prediction estimate for the perfect-foresight state $\hat{x}_{t+2|t}^*$ is obtained as:

$$\hat{x}_{t+2|t}^* = [0 \quad I]^T\hat{z}_{t+1|t} \tag{16}$$

which can therefore be used to recursively solve the RE model (1)-(2), given initial conditions knowledge only.

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[5]Since the state equation (8) is to be linearized around a (supposedly) unbiased filtering estimate of the perfect-foresight state only, the corresponding output should be used accordingly.

[6]$D_i(\Phi)(\bar{i})$ denotes the Jacobian of Φ with respect to $i = x^*, v$ evaluated at some $\bar{i}$.

[7]See Carravetta *et al.* (2002) and Lipster and Shiryaev (2004) on this issue. Note that the matrix pair (A, B) is controllable whereas the matrix pair $(A, \bar{C})$ is observable.

4. Concluding remarks

Solving stochastic RE models means finding an expression for the unobservable expectational term as a (possibly nonuniquely determined) function of the conditioning information set. Carravetta and Sorge (2010) develop a recursive algorithm, based upon classical Kalman filtering and stochastic control theory, for the solution of linear time-varying RE models with past expectations and no predetermined variables. In this note, we extend their approach to a class of nonlinear dynamic stochastic models by means of an extended Kalman filtering technique.

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