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Monotone comparative statics with separable objective functions

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Abstract

The Milgrom-Shannon single crossing property is essential for monotone comparative statics of optimization problems and noncooperative games. This paper formulates conditions for an additively separable objective function to satisfy the single crossing property. One component of the objective function is assumed to allow a monotone concave transformation with increasing differences, and to be nondecreasing in the parameter variable. The other component is assumed to exhibit increasing differences, and to be nonincreasing in the choice variable. As an application, I prove existence of an isotone pure strategy Nash equilibrium in a Cournot duopoly with logconcave demand, affiliated types, and nondecreasing costs.

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1. Introduction

Monotone comparative statics has proven to be an extremely useful tool in numerous economic applications. Fundamental concepts of cardinal supermodularity (Topkis 1978, Milgrom and Roberts 1990) have been refined over time into more general, ordinal variants (Milgrom and Shannon 1994). However, ordinal techniques are less straightforward to apply when the problem is characterized by a separable objective function. For instance, while the sum of two supermodular functions is again supermodular, the sum of two logsupermodular functions need not be logsupermodular.[1]

The present paper addresses this problem by offering conditions under which an additively separable objective function becomes eligible for the ordinal approach to monotone comparative statics. For this, the first term of the objective function is assumed to possess a monotone concave transformation with increasing differences, and to be weakly increasing in the parameter variable. The other term is assumed to exhibit increasing differences, and to be weakly decreasing in the choice variable. I will show that with these assumptions in place, the objective function indeed satisfies the Milgrom-Shannon single crossing property.

To see the theorem at work, consider a firm operating in a homogeneous good market by setting an output level (the choice variable). Assume inverse demand to be logconcave and monotone declining, and costs to be monotone increasing. For convenience, let the natural order on the set of rivals' joint output (the parameter variable) be reversed. Then, log-revenues exhibit increasing differences, and revenues are weakly increasing in the parameter variable. Moreover, the negative cost term trivially exhibits increasing differences, and is weakly decreasing in the choice variable. Thus, profits satisfy the single crossing property, and the set of best responses is monotone nondecreasing in the parameter variable.[2]

The rest of the paper is structured as follows. Elements of the Milgrom-

[1] See, e.g., Athey (2002). For definitions of standard terms, see Section 2 of the present paper.

[2] This particular setting has been studied before by Amir (1996). For an application that yields new results, see Section 5.

Shannon theory are reviewed in Section 2. Section 3 defines and characterizes the notion of concavely increasing differences. The main result of the paper is stated and proved in Section 4. Section 5 contains an application to Bayesian Cournot games.

2. Review of standard theory

This section briefly reviews the main elements of the ordinal approach to monotone comparative statics. For an introduction to this theory, see Milgrom and Shannon (1994).

A set of decisions, X, is equipped with a binary relation $\geq$. The relation $\geq$ is *reflexive* if $x \geq x$ for every $x \in X$, *transitive* if $x \geq x'$ and $x' \geq x''$ implies that $x \geq x''$ for all $x, x', x'' \in X$, and *antisymmetric* if $x \geq x'$ and $x' \geq x$ implies that $x = x'$ for all $x, x' \in X$. A set X with a reflexive, transitive, and antisymmetric binary relation $\geq$ is a *partially ordered set.* For $x, x' \in X$, denote by $x \wedge x'$ the *least upper bound*, if it exists, and by $x \vee x'$ the *greatest lower bound*, if it exists. If for any pair $x, x' \in X$, both $x \wedge x'$ and $x \vee x'$ exist, then the partially ordered set X is a *lattice.* If for any $x \neq x'$, either $x \leq x'$ or $x \geq x'$, then the lattice X is a *chain.*

Let X be a lattice, T be a partially ordered set of parameter values, and $f : X \times T \to \mathbb{R}$. Then f has *increasing [decreasing] differences* in (x, t) if for $x \geq x'$, $f(x, t) - f(x', t)$ is nondecreasing [nonincreasing] in t. The function f satisfies the *single crossing property* in $(x; t)$ if for $x' > x''$ and $t' > t''$, $f(x', t'') > f(x'', t'')$ implies that $f(x', t') > f(x'', t')$ and $f(x', t'') \geq f(x'', t'')$ implies that $f(x', t') \geq f(x'', t')$. If $f(x', t'') \geq f(x'', t'')$ implies that $f(x', t') > f(x'', t')$, then f satisfies the *strict single crossing property* in $(x; t)$. f satisfies the *dual single crossing property* in $(x; t)$ if $-f$ satisfies the single crossing property in $(x; t)$.

A function $f : X \to \mathbb{R}$ on some lattice X is *supermodular* if $f(x \wedge x') + f(x \vee x') \geq f(x) + f(x')$ for any $x, x' \in X$. The function f is *submodular* if $-f$ is supermodular. A strictly positive function is *logsupermodular* (*logsubmodular*) if the log of that function is supermodular (submodular). f is *quasisupermodular* if for any $x, x' \in X$, $f(x) \geq f(x \wedge x')$ implies that

$f(x \vee x') \geq f(x')$ and $f(x) > f(x \wedge x')$ implies that $f(x \vee x') > f(x')$. Any supermodular function is quasisupermodular. Moreover, every function on a chain is quasisupermodular.

For X a lattice, and subsets $Y, Z \subseteq X$, write $Z \leq_s Y$ if for every $z \in Z$ and every $y \in Y$, $y \wedge z \in Z$ and $y \vee z \in Y$. Given a partially ordered set T, a set-valued function M mapping elements of T to subsets of X is *monotone nondecreasing* if $t \leq t'$ implies $M(t) \leq_s M(t')$. The usefulness of the single crossing property is mainly due to the following fact, a proof of which can be found in Milgrom and Shannon (1994).

Theorem 1. (Monotonicity Theorem) *Let $f : X \times T \to \mathbb{R}$, where X is a lattice, T is a partially ordered set, and $S \subseteq X$. Then $\arg\max_{x \in S} f(x,t)$ is monotone nondecreasing in (t, S) if and only if f is quasisupermodular in x and satisfies the single crossing property in $(x; t)$.*

3. Concavely increasing differences

This section introduces the notion of concavely increasing differences and offers a characterization that will be useful to prove the main result of this paper.

Definition 1. *Let X and T be partially ordered sets. A function $g : X \times T \to \mathbb{R}$ has [strict]* ***concavely increasing differences in*** *(x, t) if for any $x' > x''$ and $t' > t''$, there exists some strictly increasing, concave transformation $\phi = \phi_{(x',x'',t',t'')}$ such that*

$$\phi(g(x', t')) - \phi(g(x'', t')) \underset{[>]}{\geq} \phi(g(x', t'')) - \phi(g(x'', t'')). \tag{1}$$

Obviously, any function with increasing differences has concavely increasing differences. Moreover, any function that is logsupermodular on the product space $X \times T$ (where both X and T are lattices) has concavely increasing differences.[3] Note, however, that there are functions that fulfill neither

[3] Another example are rootsupermodular functions, as defined by Eeckhout and Kircher (2010).

x	t	
	0	1
0	27	29
1	18	19

Example 1 – The tabulated function *g* fails to satisfy increasing differences in *(x,t)* and logsupermodularity, but has concavely increasing differences in *(x,t)*.

property and still satisfy Definition 1. Indeed, in Example 1, the inequality $\phi(19) - \phi(29) \geq \phi(18) - \phi(27)$ fails for transformations $\phi(y) = y$ and $\phi(y) = \ln y$, yet holds for $\phi(y) = -1/y$.

These examples might suggest that Definition 1 just requires g to be a function for which some concave and monotone transform has increasing differences. However, the definition is more flexible since the transformation ϕ may vary with the quadruple (x', x'', t', t'').

To prepare the key result, I state the following characterization of concavely increasing differences.

Lemma 1. *A function g has [strict] concavely increasing differences in (x, t) if and only if for any $x' > x''$ and $t' > t''$ such that $\min\{g(x'', t'), g(x', t'')\} \geq \min\{g(x', t'), g(x'', t'')\}$, the inequality*

$$g(x', t') - g(x'', t') \underset{[>]}{\geq} g(x', t'') - g(x'', t'') \tag{2}$$

holds.

Proof. The proof is given for the case of nonstrict differences only. The other case is analogous. "Only if". Assume g has concavely increasing differences in $(x; t)$, and let $x' > x''$ and $t' > t''$. Write $a = g(x'', t'')$, $b = g(x'', t')$, $c = g(x', t'')$, and $d = g(x', t')$. Then there is a strictly increasing, concave

transformation $\phi = \phi_{(x',x'',t',t'')}$ such that $\phi(d) - \phi(b) \geq \phi(c) - \phi(a)$. I wish to show that this implies $d - b \geq c - a$ provided that $\min\{b, c\} \geq \min\{a, d\}$. Since the roles of a and d in this claim are interchangable, and equally those of b and c, one may assume without loss of generality that $a \leq b \leq c$. Then necessarily $c \leq d$, because otherwise $\phi(d) < \phi(c)$ and $\phi(b) \geq \phi(a)$, in conflict with $\phi(d) - \phi(b) \geq \phi(c) - \phi(a)$. Thus, $a \leq b \leq c \leq d$. Now if one had $d - b < c - a$, then with a concave and strictly increasing ϕ, $\phi(d) - \phi(b) < \phi(c) - \phi(a)$, a contradiction. Therefore, indeed, $d - b \geq c - a$.
"If". Using the same notation, assume g satisfies $d - b \geq c - a$ provided that $\min\{b, c\} \geq \min\{a, d\}$. I need to find a strictly increasing, concave $\phi = \phi_{(x',x'',t',t'')}$ such that $\phi(b)+\phi(c) \leq \phi(a)+\phi(d)$. Consider first $\min\{b, c\} < \min\{a, d\}$. If $b = c$, the claim follows for any strictly increasing ϕ. Therefore, without loss of generality, $b < a, c, d$. In this case, define $\phi(a) = a$, $\phi(c) = c$, $\phi(d) = d$, and $\phi(b)$ sufficiently negative so that $\phi(b) + \phi(c) \leq \phi(a) + \phi(d)$. Clearly, ϕ can be extended to a strictly increasing, concave function on $\mathbb{R}$. Consider now $\min\{b, c\} \geq \min\{a, d\}$. Then, by assumption, $b + c \leq a + d$. Clearly, in this case, ϕ can be chosen linear. $\square$

Thus, concavely increasing differences requires increasing differences only when "off-diagonals" $g(x'', t')$, $g(x', t'')$ do not fall below the minimum of the "diagonals" $g(x', t')$, $g(x'', t'')$.

Lemma 1 implies that any monotone function, increasing or decreasing, that exhibits concavely increasing differences must have increasing differences. Note, however, that in typical applications the objective function is not monotone in the choice variable.

4. Separable objective functions

The main result of the paper is the following.

Theorem 2. *Let X and T be partially ordered sets. Consider functions $g, h : X \times T \to \mathbb{R}$. Assume that g has concavely increasing differences in (x, t) and is nondecreasing [nonincreasing] in t. Assume also that h has increasing differences in (x, t) and is nonincreasing [nondecreasing] in x.*

Then $g + h$ satisfies the single crossing property in $(x; t)$. If, in addition, g has strict concavely increasing differences in (x, t) and is strictly increasing [decreasing] in t, then $g + h$ satisfies the strict single crossing property in $(x; t)$.

Proof. According to the definition, $f = g + h$ satisfies the single crossing property in $(x; t)$ if for $x' > x''$ and $t' > t''$, $f(x', t'') \geq f(x'', t'')$ implies that $f(x', t') \geq f(x'', t')$ and $f(x', t'') > f(x'', t'')$ implies that $f(x', t') > f(x'', t')$. So take arbitrary $x' > x''$ and $t' > t''$. Impose

$$f(x', t'') \geq f(x'', t''). \tag{3}$$

Since h is nonincreasing in x, inequality (3) implies $g(x', t'') \geq g(x'', t'')$. Moreover, g is nondecreasing in t, so $g(x'', t') \geq g(x'', t'')$. By assumption, g has concavely increasing differences in (x, t). Thus, by Lemma 1,

$$g(x', t') - g(x'', t') \geq g(x', t'') - g(x'', t''). \tag{4}$$

But h has increasing differences in (x, t), so that

$$h(x', t') - h(x'', t') \geq h(x', t'') - h(x'', t''). \tag{5}$$

Adding (4) and (5) term by term yields

$$f(x', t') - f(x'', t') \geq f(x', t'') - f(x'', t''). \tag{6}$$

Combining this with (3), one obtains

$$f(x', t') - f(x'', t') \geq 0, \tag{7}$$

as desired. Moreover, if inequality (3) holds strictly, so does (7). This proves the claim for nonstrict differences. To prove the claim also for strict differences, note that inequality (4) is then strict, so that inequality (3) implies the strict version of (7), as required by the strict single crossing property. □

For intuition, focus on T and X being two-element subsets of $\mathbb{R}$, and ϕ being the logarithm. Clearly, the conclusion is obvious when g has actually increasing differences. So assume that the slope $\frac{g(x',t)-g(x'',t)}{x'-x''}$, regarded as a

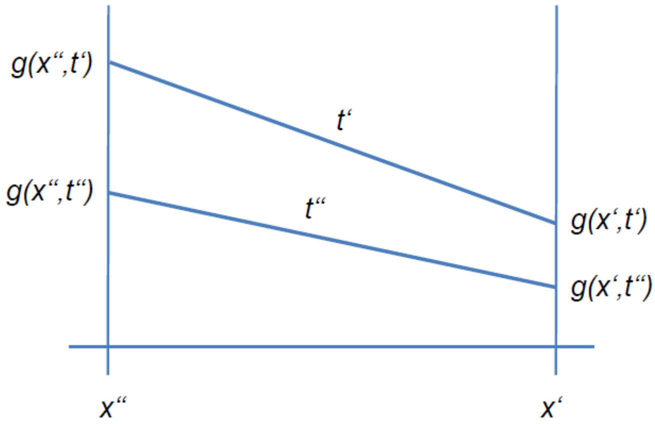

Figure 1 – If the slope of *g* with respect to *x* strictly decreases in *t*, the ratio *g(x',t)/g(x",t)* weakly increases in *t*, and *g* is monotone increasing in *t*, then *g* must be downwards sloping at *t"*.

function of t, strictly decreases, while the ratio $\frac{g(x',t)}{g(x'',t)}$ weakly increases in t, as illustrated in Figure 1. Since g is nondecreasing in t, a moment's reflection shows that this is possible only when g is strictly downward-sloping at t''.[4] But then, adding a function h that is nonincreasing in x implies the single crossing property for the sum.

To extend Theorem 2, re-order T (or, equivalently, X). E.g., assume that g has concavely decreasing differences in (x,t), which is defined in analogy to Definition 1, and that g is nonincreasing [nondecreasing] in t. Then, with h having decreasing differences in (x,t) and being nonincreasing [nondecreasing] in x, it follows that $g+h$ satisfies the dual single crossing property in $(x;t)$.

Another extension assumes that g has convexly increasing or decreasing differences in (x,t), where again, the notions are defined in analogy to Definition 1. For instance, when g has convexly increasing differences in (x,t) and is nondecreasing [nonincreasing] in t, and h has increasing differences in (x,t) and is nondecreasing [nonincreasing] in x, then $g+h$ satisfies the single

[4] Indeed, if g were upwards sloping or flat at t'', then the strictly lower slope at t' would make the ratio $g(x',t)/g(x'',t)$ decline strictly in t.

crossing property.[5]

In applications, one typically has to check also that the objective function is quasisupermodular in the decision variable.[6] Sufficient conditions for an additively separable function to be quasisupermodular in the choice variable are obviously either that (i) each term is supermodular in the choice variable, or that (ii) the choice set is a chain (e.g., a subset of $\mathbb{R}$).

With these remarks in mind, Theorem 2 can be readily applied to the comparative statics of optimization problems and noncooperative games characterized by separable objective functions. A simple example has already been given in the Introduction. The next section offers a more elaborate application.

5. Application: Bayesian Cournot games

This section deals with equilibrium existence in the undifferentiated Cournot model with affiliated types. A pure strategy Nash equilibrium is known to exist regardless of distributional assumptions provided certainty payoffs are submodular in firms' actions (Vives 1990). Under additional complementarities between actions and types, even an isotone equilibrium exists for affiliated types (Athey 2001).[7] However, cardinal submodularity in actions is not a completely innocuous assumption in the Bayesian Cournot model because uncertainty then tends to generate negative prices (cf. Einy *et al.* 2010).[8] This problem can be circumvented using Theorem 2. As will be shown now, there exists a set of simple conditions, including logconcave inverse demand and weakly increasing costs, under which an isotone pure-strategy Nash equilibrium exists in a duopoly with affiliation.

[5] Indeed, it is not difficult to check that g has convexly increasing differences in (x, t) if and only if $-g$ has concavely decreasing differences in (x, t), so that the claim follows from the first extension.

[6] See, e.g., Theorem 1.

[7] See also McAdams (2003), Van Zandt and Vives (2007), and Reny (2009).

[8] Indeed, Cournot profits that are submodular imply Novshek's (1985) marginal revenue condition on inverse demand. The marginal revenue condition, in turn, can be seen to be equivalent to inverse demand being a concave function of log-output. Hence, if inverse demand is declining somewhere, it must eventually cause negative prices.

Inverse demand is given by a nonincreasing function p, assumed to be nonnegative, nonconstant, and logconcave.[9] There are two firms $i = 1, 2$, each receiving a private signal t_i, referred to as the firm's *type*, and drawn from a compact interval $T_i \subset \mathbb{R}$. Types are *inversely affiliated*, i.e., jointly distributed according to some logsubmodular density on $T_1 \times T_2$. Each firm i produces output $x_i \geq 0$ at costs $C_i(x_i, t)$, where $t = (t_i, t_j)$ with $j \neq i$. Costs are assumed nondecreasing and continuous in output. Moreover, marginal costs are nonincreasing in own type and nondecreasing in the other firm's type. Denote firm i's strategy by $\xi_i = \xi_i(t_i)$. Expected profits of a firm i of type t_i producing output x_i read

$$f_i(x_i, t_i) = E[x_i p(x_i + \xi_j(t_j)) - C_i(x_i, t)|t_i]. \tag{8}$$

It is claimed that f_i satisfies the single crossing property in $(x_i; t_i)$ provided ξ_j is monotone increasing. For this, write $f_i = g_i + h_i$, where $g_i(x_i, t_i) = E[x_i p(x_i + \xi_j(t_j))|t_i]$ and $h_i(x_i, t_i) = -E[C_i(x_i, t)|t_i]$, respectively, denote expected revenues and (negatively signed) expected costs. To apply Theorem 2, note that ex-post revenues $x_i p(x_i + x_j)$ are logsubmodular in (x_i, x_j). Hence, because ξ_j is monotone increasing, $x_i p(x_i + \xi_j(t_j))$ is logsubmodular in (x_i, t_j). Therefore, with inversely affiliated types, g_i is logsupermodular in (x_i, t_i).[10] Moreover, as p is nonincreasing, ξ_j is monotone increasing, and types are inversely affiliated, it follows that g_i is nondecreasing in t_i. Consider now the cost term. By assumption, $C_i(x_i, t)$ is submodular in (x_i, t_i) and supermodular in (x_i, t_j). As types are inversely affiliated, it follows that h_i is supermodular in (x_i, t_i).[11] Furthermore, since costs are nondecreasing in output, h_i is nonincreasing in x_i. Thus, f_i satisfies the single crossing property in $(x_i; t_i)$ for any nondecreasing ξ_j. Moreover, under the assumptions made on inverse demand, revenue is declining when own output exceeds the point of unitary elasticity. Hence, there is an output level above which no firm has an incentive to operate. It follows now from Corollary 2.1 in Athey (2001) that an isotone pure strategy Nash equilibrium exists.

[9] Logconcavity of p requires that $\log p$ is concave on the interval where $p > 0$.
[10] See the discussion following Lemma 2 in Athey (2002).
[11] Cf. Fact (v) in Athey (2001, p. 872).

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