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### Ramsey Fiscal Policy and Endogenous Growth: A Comment

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#### Abstract

Recently, Park (2009, *Economic Theory* 39, 377--398) extended the Barro endogenous growth model (1990) by assuming that tax rate is optimally chosen by the government and labor supply is elastic. Park claimed to have proved the existence of multiple balanced growth paths that exhibit zero growth rate and local indeterminacy. In this comment, it is shown that his claim is incorrect. The model has a unique balanced growth path that may exhibit positive growth, and the model has no transitional dynamics.

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# 1 Introduction

Recently, Park (2009) extended the Barro endogenous growth model (1990) by assuming that tax rate is optimally chosen by the government and labor supply is elastic. Park claimed to have proved the existence of multiple balanced growth paths that exhibit zero growth rate and local indeterminacy. In this comment, it will be shown that his claim is incorrect: the model has a unique balanced growth path that may exhibit positive growth with no transitional dynamics.

The remainder of this note is organized as follows. In Section 2, we summarize the Park (2009) model. In Section 3, we formulate the (Ramsey) optimal policy problem and summarize the conditions that characterize an equilibrium. The main findings are summarized and discussed in Section 4. Finally, conclusions are presented in Section 5.

## 2 The Model

The Park (2009) model can be summarized as follows. The representative household in the model economy has the preferences:

$$\int_0^{\infty} \left[ \log(x) - \frac{1}{1+\varepsilon} l^{1+\varepsilon} \right] e^{-\rho t} dt, \quad \varepsilon > 0, \quad (1)$$

where  $x$  is consumption, and  $l$  is labor supply. The household seeks to maximize its lifetime utility subject to the budget constraint:

$$\dot{k} = (1 - \tau)(wl + rk) + \pi,$$

where  $k$  is capital;  $w$  and  $r$  denote wage and rental rates, respectively;  $\tau$  is the income tax rate; and  $\pi$  is firm profits.

The representative firm produces final goods  $y$  via the technology

$$y = f(k, l, g) = Ag^{1-\alpha} k^{\alpha} l^{\beta}, \quad (2)$$

where  $\beta = 1 - \alpha$ . In (2),  $g$  denotes productive government spending, which is taken as given by individuals. The firm chooses  $k$  and  $l$  optimally to maximize its profit, denoted by  $\pi = f(k, l, g) - wl - rk$ . The markets for factors and goods are perfectly competitive.

The government implements fiscal policies  $\tau$  and  $g$  while keeping the budget balanced. Thus, if the government chooses a tax rate  $\tau$ , the government spending  $g$  is determined by

$$g = \tau(wl + rk). \quad (3)$$

The model is closed using the market-clearing condition

$$\dot{k} = y - x - g. \quad (4)$$

Given a tax rate  $\tau$ , the equilibrium allocation of  $k$ ,  $x$ , and  $l$  can be characterized by

equations (15a)–(15c) in Park (2009), which are reproduced below:

$$\dot{k} = \eta(\tau)kl^{\beta/\alpha} - x, \quad (5a)$$

$$\dot{x} = x [\alpha\eta(\tau)l^{\beta/\alpha} - \rho], \quad (5b)$$

$$l = \left[ \beta\eta(\tau)\frac{k}{x} \right]^{1/(1+\varepsilon-\beta/\alpha)}, \quad (5c)$$

where  $\eta(\tau) \equiv A^{1/\alpha}(1-\tau)\tau^{(1-\alpha)/\alpha}$ . The associated transversality condition (TVC) can be expressed as  $\lim_{t \rightarrow \infty} e^{-\rho t}k(t)/x(t) = 0$ . The above equations are derived from (3), (4), and the optimality conditions to the household and firm optimization problems, with  $g$ ,  $w$ , and  $r$  being eliminated. Detailed derivations are omitted in the interest of space; readers are referred to Park (2009).

### 3 The Ramsey Problem

The government sets the income tax rate optimally in the spirit of Ramsey (1927). It chooses a path of  $\tau$  such that the competitive equilibrium resulting from this tax policy maximizes the household lifetime utility. The Ramsey problem can be formulated as if the government chooses paths of  $\tau$ ,  $x$ ,  $l$ , and  $k$  to maximize (1) subject to the equilibrium conditions (5a)–(5c).

While the Ramsey problem is correctly stated, the analysis by Park (2009) becomes inaccurate when the control theory is applied to solve the problem. Park substitutes (5c) into (5a) to yield

$$\dot{k} = k [1 - \beta l^{-(1+\varepsilon)}] \eta(\tau)l^{\beta/\alpha} \quad (5a')$$

and then sets up a Hamiltonian (equation 17 in Park, 2009) that incorporates (5b) and (5a'):

$$H = \log(x) - \frac{l^{1+\varepsilon}}{1+\varepsilon} + \lambda_x x [\alpha\eta(\tau)l^{\beta/\alpha} - \rho] + \lambda_k k [1 - \beta l^{-(1+\varepsilon)}] \eta(\tau)l^{\beta/\alpha},$$

where  $\lambda_x$  and  $\lambda_k$  are multipliers. This Hamiltonian is incorrect because it omits (5c). Equation (5c) can be omitted if it can be implied by (5b) and (5a'), or if one of the variables in the Ramsey problem has been eliminated, but neither condition is met. Based on the Hamiltonian, the results obtained and summarized in Park's Propositions 1–4 are incorrect. In the remainder of this paper, we reinvestigate the Ramsey problem and correct Park's Propositions.

We first substitute (5c) into (5a) and (5b) to obtain

$$\dot{k}/k = \beta^{\phi-1}\eta(\tau)^\phi(k/x)^{\phi-1} - x/k, \quad (5a^*)$$

$$\dot{x}/x = \alpha\beta^{\phi-1}\eta(\tau)^\phi(k/x)^{\phi-1} - \rho. \quad (5b^*)$$

We then substitute (5c) into the utility function to eliminate the variable  $l$ , and set up the

following Hamiltonian:

$$H = \log(x) - \frac{(\beta\eta(\tau)k/x)^\phi}{1 + \varepsilon} + \lambda_k k [\beta^{\phi-1}\eta(\tau)^\phi(k/x)^{\phi-1} - x/k] + \lambda_x x [\alpha\beta^{\phi-1}\eta(\tau)^\phi(k/x)^{\phi-1} - \rho],$$

where  $\lambda_k$  and  $\lambda_x$  are multipliers associated with (5a\*) and (5b\*), and

$$\phi \equiv \frac{1 + \varepsilon}{1 + \varepsilon - \beta/\alpha}.$$

Note that  $1 + \varepsilon - \beta/\alpha > 0$  and  $\phi > 1$  according to the assumptions in Park (2009). The first-order conditions with respect to  $\tau$ ,  $k$ , and  $x$  can be summarized as:

$$0 = \left[ \frac{\beta\eta(\tau)k}{x} \right]^{\phi-1} \left[ \lambda_k k + \alpha\lambda_x x - \frac{\beta k/x}{1 + \varepsilon} \right] \phi\eta'(\tau), \quad (6a)$$

$$\dot{\lambda}_k = \rho\lambda_k - \lambda_k \left( \frac{\dot{k}}{k} \right) - \frac{\lambda_k x}{k} + \frac{\eta(\tau)}{k} \left[ \frac{\beta\eta(\tau)k}{x} \right]^{\phi-1} \left[ (1 - \phi)(\lambda_k k + \alpha\lambda_x x) + \frac{\phi\beta k/x}{1 + \varepsilon} \right], \quad (6b)$$

$$\dot{\lambda}_x = \rho\lambda_x - \lambda_x \left( \frac{\dot{x}}{x} \right) - \frac{1}{x} + \lambda_k - \frac{\eta(\tau)}{x} \left[ \frac{\beta\eta(\tau)k}{x} \right]^{\phi-1} \left[ (1 - \phi)(\lambda_k k + \alpha\lambda_x x) + \frac{\phi\beta k/x}{1 + \varepsilon} \right], \quad (6c)$$

and the TVCs,  $\lim_{t \rightarrow \infty} e^{-\rho t} \lambda_x x = \lim_{t \rightarrow \infty} e^{-\rho t} \lambda_k k = 0$ .

In the above system, consumption  $x$  is treated as a state variable. Yet, the initial value of  $x$  is not predetermined. The literature (e.g., Chamley, 1986; Cohen and Michel, 1988) suggests that the shadow price of a state variable that has a free initial value is predetermined and initially zero. Thus, we impose an initial condition on the shadow price of  $x$ :

$$\lambda_x(0) = 0. \quad (7)$$

This initial condition is not imposed in Park (2009). As will be shown later, this condition is important for examining the stability of a steady state; a lack of it may lead to spurious stability.

In summary, the Ramsey problem is to be solved by finding a solution to the system of (5a\*), (5b\*), (6a)–(6c), (7), and the TVCs.

## 4 The Results

From (6a) we can immediately obtain that the optimal tax rate, denoted by  $\tau^*$ , must satisfy  $\eta'(\tau^*) = 0$ .<sup>1</sup> Thus,  $\tau^* = 1 - \alpha$ , at which the function  $\eta$  has a maximum. Notably, the tax rate  $\tau^*$  coincides with the optimal tax rate in the Barro model (1990), where labor supply is inelastic. As pointed out by Barro and Sala-i-Martin (1992), the efficiency condition for the government spending requires the marginal product of  $g$  to equal its cost, i.e.,  $\partial y/\partial g = 1$ . Thus,  $g/y = \tau = 1 - \alpha$  as implied by (2). Therefore, under the Cobb-Douglas assumption, the elasticity of labor supply has no role in determining the optimal tax rate.

<sup>1</sup>A proof for  $\lambda_k k + \alpha\lambda_x x - \beta(k/x)/(1 + \varepsilon) \neq 0$  is omitted, but it is available upon request.

By the fact that  $\tau$  equals the constant  $\tau^*$ , (5a\*), (5b\*), (6b), and (6c) represent a dynamic system of  $k$ ,  $x$ ,  $\lambda_k$ , and  $\lambda_x$ . In order to explore the steady state, the system is transformed to induce stationarity. Define  $z = x/k$ ,  $\nu_k = \lambda_k k$ , and  $\nu_x = \lambda_x x$ . Subtracting (5a\*) from (5b\*) and using the fact that  $\tau = \tau^* = 1 - \alpha$ , we obtain that

$$\frac{\dot{z}}{z} = z - Bz^{1-\phi} - \rho, \quad (8)$$

where we have defined

$$B \equiv (\beta\bar{\eta})^\phi, \quad \bar{\eta} \equiv \eta(\tau^*) = A^{1/\alpha}\alpha(1-\alpha)^{(1-\alpha)/\alpha}.$$

Multiplying (6b) by  $k$  and (6c) by  $x$ , we obtain that

$$\begin{aligned} \dot{\nu}_k &= \rho\nu_k - \nu_k z + \frac{Bz^{1-\phi}}{\beta} \left[ (1-\phi)(\nu_k + \alpha\nu_x) + \frac{\phi\beta}{(1+\varepsilon)z} \right], \\ \dot{\nu}_x &= \rho\nu_x - 1 + \nu_k z - \frac{Bz^{1-\phi}}{\beta} \left[ (1-\phi)(\nu_k + \alpha\nu_x) + \frac{\phi\beta}{(1+\varepsilon)z} \right]. \end{aligned} \quad (9)$$

Combining the last two equations yields the differential equation:

$$\frac{d}{dt}(\nu_k + \nu_x) = \rho(\nu_k + \nu_x) - 1.$$

This differential equation can be easily solved, and the only solution satisfying the TVCs is  $\nu_k + \nu_x = 1/\rho$ .<sup>2</sup> Using this result, (9) can be rewritten as

$$\dot{\nu}_x = [\rho + (1-\phi)Bz^{1-\phi} - z]\nu_x - \frac{(1-\phi)Bz^{1-\phi}}{\beta\rho} - \frac{\phi Bz^{-\phi}}{1+\varepsilon} + \frac{z-\rho}{\rho}. \quad (10)$$

The system is now reduced to equations (8) and (10).

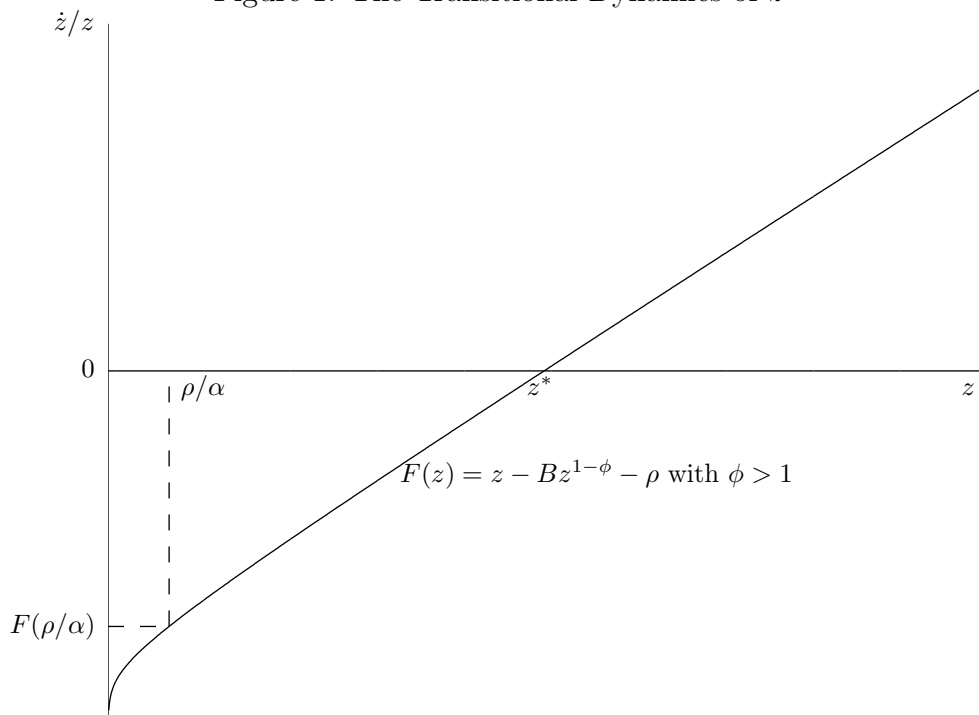
On a balanced growth path,  $\dot{z} = \dot{\nu}_x = 0$ . If  $F(z)$  stands for the right-hand side of (8), the steady state of  $z$  solves  $F(z) = 0$ . As  $\phi > 1$ ,  $F$  is strictly increasing and concave with  $\lim_{z \rightarrow 0} F(z) = -\infty$  and  $\lim_{z \rightarrow \infty} F(z) = \infty$ . Thus, as shown in Figure 1, there exists a unique steady state  $z^*$ . Using the fact that  $Bz^{*(1-\phi)} = z^* - \rho$ , the steady state of  $\nu_x$  can be denoted by  $\nu_x^*$  and obtained from (10):

$$\nu_x^* = \frac{\phi + \beta - 1}{\beta\rho\phi} - \frac{1}{(1+\varepsilon)z^*}.$$

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<sup>2</sup>Equivalently,  $\lambda_k k + \lambda_x x = 1/\rho$ . This last result implies that one of the four variables  $k$ ,  $x$ ,  $\lambda_k$ , and  $\lambda_x$  can be expressed in terms of the other three variables. Thus, the original system of (5a\*), (5b\*), (6b), and (6c) can be represented alternatively as a  $3 \times 3$  system.

Figure 1: The Transitional Dynamics of  $z$



The steady-state growth rate can read off from (5b\*):

$$\gamma \equiv \frac{\dot{x}}{x} = \alpha\beta^{-1}(\beta\bar{\eta})^\phi z^{*1-\phi} - \rho = \alpha\beta^{-1}Bz^{*1-\phi} - \rho = \alpha\beta^{-1}\left(z^* - \frac{\rho}{\alpha}\right).$$

The growth rate  $\gamma$  is positive if and only if  $z^* > \rho/\alpha$ . As shown in Figure 1, to ensure that  $z^* > \rho/\alpha$ , we require:

$$F(\rho/\alpha) = \rho/\alpha - B(\rho/\alpha)^{1-\phi} - \rho < 0. \quad (11)$$

Finally, we turn to transitional dynamics. Since  $F(z)$  is upward sloping, the steady state  $z^*$  is unstable. The only path of  $z(t)$  that satisfies the TVCs is  $z(t) = z^*$  for all  $t$ . Thus, like an *AK* model, the present model exhibits no transitional dynamics of real variables. Given that  $z(t) = z^*$ , (10) can be rewritten as  $\dot{\nu}_x = -\phi(z^* - \rho)(\nu_x - \nu_x^*)$ ; the solution is

$$\nu_x(t) = \nu_x^* + (\nu_x(0) - \nu_x^*)e^{-\phi(z^* - \rho)t}.$$

If we maintain the assumption (11),  $z^* > \rho/\alpha > \rho$ . Hence,  $\phi(z^* - \rho) > 0$ , and the last solution for  $\nu_x$  is convergent regardless of  $\nu_x(0)$ . This may lead one to conclude that the solution is indeterminate. However, the indeterminacy is spurious. As implied by (7), the initial value  $\nu_x(0)$  is given by  $\lambda_x(0)x(0) = 0$ . Thus, the solution for  $\nu_x$  is unique. Since only the auxiliary variable  $\nu_x$  exhibits transitional dynamics, we conclude that the model has no transitional dynamics as far as real variables are concerned.

## 5 Concluding Remarks

In this note, we have reinvestigated the Ramsey fiscal policies in the Park (2009) model, and we have corrected Park's Propositions by showing that the model has a unique steady state that may exhibit positive growth with no transitional dynamics.

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