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# A note on continuous time models with general cash-in-advance constraints

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## Abstract

This note completely characterizes continuous time models with general cash-in-advance constraints in that money is demanded for purchasing not only consumption goods but also for making all or some investments. Examining the three-dimensional dynamics of an exogenous growth model with general cash-in-advance constraints is unique. Comparative static analysis shows that increased inflation or a strengthened cash-in-advance constraint lowers the level of the capital stock in the long run. We also show that the steady state is locally stable.

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# 1 Introduction

In a cash-in-advance economy, money is demanded for purchasing goods. Stockman (1981) compared two types of cash-in-advance models in a discrete time framework: in one, cash-in-advance constraints apply only to consumption goods; in the other, such constraints apply to both consumption and investment goods. While Stockman focused on the steady state, Abel (1985) examined the dynamic properties of the two models. Palivos et al. (1993), having extended cash-in-advance models by letting money be demanded for purchasing a fraction of the capital goods as well as consumption goods, successfully explained fluctuations in the velocity of money. However, because they used dynamic programming, they did not examine stability properties and, for analytical tractability, they assumed a depreciation rate of 100%.

Although describing dynamics in a continuous time framework is expected to be easier than doing so in a discrete time framework, only Chen and Guo (2008) have so far developed a dynamic general cash-in-advance model. Wang and Yip (1992) considered a general cash-in-advance model with endogenous labor, but they focused on comparative statics. Mansoorian and Mohsin (2004) analyzed the local dynamics of a cash-in-advance model incorporating endogenous labor, but their constraint applies only to purchasing consumption goods. Because many studies of continuous time cash-in-advance models do not establish whether their continuous-time translation is consistent with their discrete time model, Kam (2004) wrote a note to justify such a transformation. Nevertheless, Kam (2004) focused on the comparative statics around the steady state of two extreme cases.

Only Chen and Guo (2008) have examined the dynamic analysis of endogenous growth by using a generalized cash-in-advance model. However, they focused on endogenous growth models having two-dimensional dynamics on the balanced growth path. To the best of our knowledge, the three-dimensional dynamics of exogenous growth models incorporating general cash-in-advance constraints remains unexamined. This is mainly because, in the literature, there are two extreme cases, in which different variables are used to characterize the dynamics.[1] Thus, one must choose an appropriate set of variables for describing generalized cash-in-advance models in a unified manner.

This note completely characterizes continuous time models with general cash-in-advance constraints in that money is demanded for purchasing not only consumption goods but also for making all or some investments. Dynamics are characterized by consumption, capital, and the shadow price of money. The note undertakes comparative static analysis with respect to the growth rate of money and the strength of the constraints, and examines the local stability properties around the steady state. We show that increased inflation or a strengthened cash-in-advance constraint lowers the level of the capital stock in the long run. We also show that the steady state is locally stable.

This note makes three major contributions to the literature. First, we succeed in describing the dynamics of models with general cash-in-advance constraints in a unified way. This could allow more general models to be built in which the degree of constraint is changing.[2] Second, to describe the unified representation, we select consumption, capital, and the shadow price of money. This

[1] For example, see Kam (2004), whose model with constraints on consumption uses consumption, capital, and the shadow price of assets (or the sum of capital and money), whereas the model with constraints on both consumption and investment uses consumption, capital, money, the shadow price of capital, and the shadow price of assets.

[2] For example, Pelvis et al. (1993) considered the degree of constraints is a function of the inflation rate and an exogenous measure of credit looseness.

selection provides an interesting contrast between our monetary model and real or other monetary models.[3] Third, for the first time, we find that the dynamic system is locally stable in the case of general constraints. Such a finding is important; it assures that the steady state is neither unstable nor generating sunspot equilibria for any degree of constraint.

## 2 The Model Economy

This section describes our model economy and derives the monetary equilibrium and the steady state. Consider a monetary economy in which consumption, the capital stock, the money stock, the price level, and real balances at period $t$ are denoted by $c_t$, $k_t$, $M_t$, $P_t$, and $m_t = M_t/P_t$, respectively. Economic agents are infinitely lived, and have perfect foresight and complete access to the capital market. Their preferences are commonly characterized by the instantaneous utility function $u(c_t)$, and agents have a constant rate of time preference $\rho$. The technology is characterized by the production function $f(k_t)$. We make the standard assumptions that $u$ and $f$ are both strictly increasing and strictly concave.

The homogeneous economic agents maximize their lifetime utility, but face two constraints. The first is the budget constraint:

$$\dot{k}_t + \dot{m}_t = f(k_t) + v_t - c_t - \pi_t m_t \tag{1}$$

where $v_t$ is a lump-sum transfer, and $\pi_t = \dot{P}_t/P_t$ is the rate of inflation. The second is the cash-in-advance constraint:

$$c_t + \Gamma \dot{k}_t \leq m_t \tag{2}$$

for $0 \leq \Gamma \leq 1$. When $\Gamma = 0$, the cash-in-advance constraint applies only to the purchase of consumption goods; when $\Gamma = 1$, the constraint indicates that money is also needed for investment. The former and latter cases, respectively, are continuous versions of those used by Lucas (1980) and Stockman (1981). The parameter $\Gamma$ represents the degree of credit tightness.

The agents choose $c_t$ and $m_t$ to maximize their lifetime utility:

$$\int_0^\infty u(c_t)\exp(-\rho t)dt \tag{3}$$

subject to the budget and the cash-in-advance constraints (1) and (2), and the initial conditions $k_0 > 0$ and $M_0 > 0$.

The government behaves in a (monetary, theoretically) conventional way. It prints money at a constant rate $\mu$ and runs a balanced budget by transferring seigniorage revenues to consumers in a lump-sum way: $v_t = \mu m_t$.

In equilibrium, the money and the goods markets clear:

$$\begin{aligned} \dot{m}_t &= (\mu - \pi_t)m_t && (4) \\ \dot{k}_t &= f(k_t) - c_t. && (5) \end{aligned}$$

A monetary equilibrium is a path involving all variables, $\{c_t, k_t, m_t, \pi_t, \xi_t, \eta_t, \zeta_t\}_{t\in[0,\infty)}$, with $\pi_t$ being positive, on which the representative agent maximizes (3) subject to (1), (2), and the initial

[3] In the money-in-the-utility models or transaction-cost models, consumption, capital, and real balances are often used to describe the dynamics.

conditions, and subject to government behavior and market clearing. The last three variables $\xi_t$, $\eta_t$, and $\zeta_t$ are defined later. Below, the time index is omitted to economize on notation.

Consider the maximization problem of the representative agent. We denote investment by $x = \dot{k}$.[4] [5] The Hamiltonian of this problem is:

$$H = u(c) + \xi(f(k) + v - c - \pi m - x) + \eta x + \zeta(m - c - \Gamma x),$$

where $\zeta$ is the Lagrange multiplier for the cash-in-advance constraints, and $\xi$ and $\eta$ are the costate variables for $\dot{m}$ and $\dot{k}$, respectively. The first-order condition yields:

$$\begin{aligned} u' &= \xi + \zeta, && (6)\\ \eta &= \xi + \zeta\Gamma, && (7)\\ \dot{\xi} - \rho\xi &= \xi\pi - \zeta, && (8)\\ \dot{\eta} - \rho\eta &= -\xi f', && (9) \end{aligned}$$

and the transversality conditions $\lim_{t\to\infty} m_t \xi_t \exp(-\rho t) = 0$ and $\lim_{t\to\infty} k_t \eta_t \exp(-\rho t) = 0$. Note that the Hamiltonian is concave with respect to $c$, $m$, $k$, and $x$ for any $\xi \geq 0$, $\eta \geq 0$, and $\zeta \geq 0$.

The costate variables $\xi$ and $\eta$ are interpreted as the shadow prices of real balances and the capital stock, respectively. When $\Gamma = 0$, equation (7) indicates that $\eta = \xi$, which implies that real balances and the capital stock have the same shadow price. An increase in $\Gamma$ raises the price of the capital stock because the cash-in-advance constraints make capital holdings more expensive. The costate variables can be interpreted as the derivatives of the indirect utility function or as the value function in the dynamic programming approach.

The multiplier $\zeta$ is the price of the cash-in-advance constraint. The marginal utility is equal to the sum of the shadow prices of the constraint and real balances. When $\zeta = 0$, the marginal utility is the shadow price of the capital stock or that of real balances for any value of $\Gamma$. We assume that the constraint is binding; i.e., $\zeta > 0$. Then, only in the case of $\Gamma = 1$ is the marginal utility equal to the shadow price of the capital stocks.

By using the time difference of (7) or $\Gamma\dot{\zeta} = \dot{\eta} - \dot{\xi}$, (8), and (9), we obtain:

$$\Gamma(\dot{\zeta} - \rho\zeta) = \zeta - \xi(f' + \pi).$$

From (6), it follows that $u''\dot{c} = \dot{\xi} + \dot{\zeta}$ and:

$$\Gamma u''(c)\dot{c} = \{\Gamma\rho + (1-\Gamma)\}u'(c) - \{(1-\Gamma)(1+\pi) + f'(c)\}\xi. \quad (10)$$

The dynamics of $c$ are presented as the dynamics of $\xi$ and $c$ given $\pi$.

Combining the cash-in-advance constraint (2) and the goods market clearing condition (5) leads to:

$$m(c,k) = c + \Gamma\dot{k} = (1-\Gamma)c + \Gamma f(k). \quad (11)$$

[4]Without loss of generality, capital depreciation is assumed to be zero. If $x = \dot{k} + \delta k$, where $\delta$ is a constant depreciation rate parameter, the first-order conditions, shown later, include $\dot{\eta} - \rho\eta = -\xi_t(f'(k) - \delta)$ in place of equation (9).

[5]In a discrete time framework, investment is often defined as $k_t - (1-\delta)k_{t-1}$ with a constant depreciation rate parameter of $\delta$. When $\delta = 1$, depreciation is 100%, and $x_t = k_t$. However, in a continuous time framework, 100% depreciation does not make investment equal to capital.

Using the time derivatives of the above equation and the money market clearing condition (4) yields:

$$\pi(c,k,\dot{c},\dot{k}) = \mu - g(c,k,\dot{c},\dot{k}), \tag{12}$$

where $g(c,k,\dot{c},\dot{k}) = \{(1-\Gamma)\dot{c}+\Gamma f'(k)\dot{k}\}/\{(1-\Gamma)c+\Gamma f(k)\}$.

By substituting (12) into (10) and (8), we obtain the equilibrium dynamics as follows:

$$\begin{aligned}
\Gamma\frac{u''(c)}{\xi}\dot{c}-(1-\Gamma)g(c,k,\dot{c},\dot{k}) &= \rho\Gamma\frac{u'(c)}{\xi}-f'(k)-(1-\Gamma)\{1+\mu-\frac{u'(c)}{\xi}\}, &(13)\\
\frac{\dot{\xi}}{\xi}+g(c,k,\dot{c},\dot{k}) &= \rho+\mu+1-\frac{u'(c)}{\xi}, &(14)
\end{aligned}$$

and (9). By using the dynamics of $c_t$, $k_t$, and $\xi_t$, we can determine those of the variables $m_t, \pi_t, \zeta_t$, and $\eta_t$ from (11), (12), (6), and (7), respectively.

When $\Gamma = 0$, equations (13) and (14) become:

$$\begin{aligned}
\frac{\dot{c}}{c} &= 1+\mu+f'(k)-\frac{u'(c)}{\xi}\\
\frac{\dot{\xi}}{\xi} &= \rho-f'(k).
\end{aligned}$$

When $\Gamma = 1$, equations (13) and (14) become:

$$\begin{aligned}
\frac{u''(c)\dot{c}}{\xi} &= \rho\frac{u'(c)}{\xi}-f'(k)\\
\frac{\dot{\xi}}{\xi}+\frac{f'(k)\dot{k}}{f(k)} &= \rho+\mu+1-\frac{u'(c)}{\xi}.
\end{aligned}$$

The steady state $(c^*,k^*,m^*,\pi^*,\xi^*,\eta^*,\zeta^*)$ is defined as the equilibrium that satisfies $\dot{c} = \dot{\xi} = \dot{k} = 0$. The variables are determined as follows. First, $\pi^* = \mu$ from (12). Then, canceling out $u'(c)/\xi$ in (13) and (14) yields:

$$f'(k^*) = \rho+\Gamma\rho(\rho+\mu). \tag{15}$$

Note that the capital stock $k^*$ is uniquely determined when $\rho+\Gamma\rho(\rho+\mu) > 0$, $\lim_{k\to 0} f'(k) = \infty$, and $\lim_{k\to\infty} f'(k) = 0$. Once $k^*$ is determined, the other variables are uniquely obtained as follows: $c^* = m^* = f(k^*)$, $\xi^* = u'(c^*)/(1+\mu+\rho)$, $\zeta^* = u'(c^*)-\xi^*$, and $\eta^* = \xi^*+\zeta^*\Gamma$.

All variables except $\pi^*$ are positive as long as $\rho+\mu > 0$. When $\rho+\mu = 0$, the cash-in-advance constraint is not binding ($\zeta^* = 0$). For $\zeta^* > 0$, the government would have to choose a money supply growth rate of $\rho+\mu > 0$.

# 3 Comparative Statics and Dynamic Analysis

This section analyzes the comparative statics, and investigates local stability properties.

First, we obtain the comparative statics. As shown in the previous section, the steady state in our model economy is determined successively. From (15), it is easy to demonstrate that at the steady state:

$$\frac{dk}{d\mu} = \frac{\Gamma\rho}{f''(k^*)} \leq 0$$
$$\frac{dk}{d\Gamma} = \frac{\rho(\rho+\mu)}{f''(k^*)} < 0.$$

The first equation shows that inflation reduces the capital stock in the long run unless $\Gamma = 0$. The second equation indicates that stronger constraints decrease the capital stock in the long run. We also find $dc/d\mu = dm/d\mu = f'dk/d\mu \leq 0$ and $dc/d\Gamma = dm/d\Gamma = f'dk/d\Gamma < 0$ at the steady state.

The mechanism underlying the comparative statics with respect to $\mu$ is as follows. From (8), at the steady state, an increased growth of money, $\mu$, raises the shadow price of cash-in-advance constraints to real balances, $\xi^*/\zeta^* = \rho + \mu$. From (7) at the steady state, an increased $\xi^*/\zeta^*$ affects the shadow price ratio of the capital stock to real balances $\eta^*/\zeta^* = 1 + \Gamma\xi^*/\zeta^*$ when $\Gamma > 0$. When $\Gamma = 0$, the shadow price ratio of capital to real balances is unchanged and, as Stockman (1981) found, capital can be freely obtained by bartering. Because (9) is $f'(k^*) = \rho\eta^*/\zeta^*$ at the steady state, an increase in $\eta^*/\zeta^*$ lowers the capital stock in the long run. When $\Gamma > 0$, increased inflation raises the cost of capital relative to real balances, and the constraint on at least some investment operates as a tax on investment goods.

Analogous are the comparative statics with respect to $\Gamma$. Given $\xi^*/\zeta^* > 0$, an increased $\Gamma$ raises the shadow price ratio of the capital stock to real balances $\eta^*/\zeta^* = 1 + \Gamma\xi^*/\zeta^*$. As shown in the previous paragraph, an increased $\eta^*/\zeta^*$ lowers the capital stock in the long run. Therefore, a strengthened cash-in-advance constraint lowers the level of the capital stock. Note that when $\rho + \mu = 0$, the nominal interest rate is zero in the long run, the constraint is no longer binding, and $\Gamma$ has no role in determining the level of the capital stock in the long run.

Next, we examine the local stability properties. The complete dynamic system for $c$, $\xi$, and $k$ is represented by (9), (13) and (14). We focus on the local stability properties. Linearizing around the steady state yields:

$$\begin{bmatrix} \dot{\xi} \\ \dot{c} \\ \dot{k} \end{bmatrix} = [B]^{-1}[A] \begin{bmatrix} d\xi^* \\ dc^* \\ dk^* \end{bmatrix},$$

where $[A]$ and $[B]$ are respectively specified at the steady state in:

$$[A] = \begin{bmatrix} \rho+\mu+1 & -u'' & 0 \\ -(1-\Gamma)(1+\mu) - f' & (1-\Gamma+\rho\Gamma)u'' & -\xi f'' \\ 0 & -1 & f' \end{bmatrix}$$
$$[B] = \begin{bmatrix} 1 & \psi & \phi \\ 0 & \Gamma u'' - (1-\Gamma)\psi & -(1-\Gamma)\phi \\ 0 & 0 & 1 \end{bmatrix},$$

where $\psi = (1-\Gamma)\xi^*/\{(1-\Gamma)c^* + \Gamma f(k^*)\} \geq 0$ and $\phi = \Gamma\xi f'(k^*)/\{(1-\Gamma)c^* + \Gamma f(k^*)\} \geq 0$. Note that the coefficients in the matrix are evaluated at the steady state.

The dynamic system of the monetary model has three dimensions with two jump variables, requiring one negative root and two positive roots in the characteristic function. For there to be only one negative root in the dynamic system, the determinant (the products of the three roots) must be negative, and the trace (the sum of the roots) must be positive.

The determinant of $[B]^{-1}[A]$ is equal to $\det[A]/\det[B]$. The determinants of $[A]$ and $[B]$ are:

$$\begin{aligned}\det[A] &= -\xi f''(\rho+\mu+1) > 0\\ \det[B] &= \Gamma u'' - (1-\Gamma)\psi < 0.\end{aligned}$$

Hence, $\det\{[B]^{-1}[A]\} < 0$. Given that:

$$[B]^{-1} = \frac{1}{\det[B]}\begin{bmatrix} \Gamma u'' - (1-\Gamma)\psi & -\psi & -\phi\Gamma u'' \\ 0 & 1 & (1-\Gamma)\phi \\ 0 & 0 & \Gamma u'' - (1-\Gamma)\psi \end{bmatrix},$$

the trace is:

$$\mathrm{tr}\{[B]^{-1}[A]\} = \frac{\Gamma u''(\rho+\mu+1) - \rho(1-\Gamma)\psi + (1-\Gamma+\rho\Gamma+\Gamma f')u''}{\det[B]} > 0.$$

Therefore, the steady state is locally stable for any value of $0 \leq \Gamma \leq 1$.

# 4 Conclusion

This note completely characterized continuous time models with general cash-in-advance constraints in that money is demanded for purchasing not only consumption goods but also for making at least some investment. Dynamics were characterized by consumption, capital, and the shadow price of money. To the best of our knowledge, our examination of the three-dimensional dynamics of an exogenous growth model with general cash-in-advance constraints is unique.

In many monetary macroeconomic models, a standard monetary shock is a stochastic change in the growth rate of the money supply. Based on our uniform characterization of general cash-in-advance models, the tightness of cash-in-advance constraints is another candidate for a monetary shock. Incorporating these two shocks into a stochastic monetary model and empirically analyzing such a model are important tasks for future research.

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