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Nash behaviour and public good

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Abstract

In this note we analyse the provision of a pure public good with non constant production cost in the context of a federation of jurisdictions with two tiers of Government: the central and the local. The central government aims at welfare maximization but this objective is constrained to the use of lump sum transfer. Local governments aim at their own utility maximization and they behave according to the Nash rule. The production cost for the public good is affected by the jurisdiction's type (high or low) and by the quantity of the good that is produced. It is shown that a social welfare improvement might take place, in some circumstances, even without any central government intervention. On the other hand a first best is unreachable under the hypothesis of Nash behaviour and lump sum transfer among jurisdictions.

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1 Introduction

Public goods are not infrequent in real world and they may be considered as a particular case of economic externality (Tresch, 2002). Public goods models are used to explain central government's economic policies, and the literature has shown much interest in examining how decentralized Nash equilibrium might approach Pareto efficiency with appropriate incentive schemes under different information requirements. Also if Williams (1966) claims that "the complex interactions that occur even in highly simplified situations make it impossible to predict a priori whether undersupply or oversupply will generally result", with perfect information the standard literature assess that when a public good is privately provided, then the level of its provision turns to be at a lower level with respect to the optimal socially desirable one. However, in the context of fiscal federalism income redistribution might be ineffective, since (Warr 1983) shows that the overall level of public good individually supplied might be independent from income redistribution. The neutrality theorem has been originally discussed by (Kemp 1984), which extends the theorem to the case of more than one public good, and by (Bergstrom et al. 1986) which "analyze the extent to which government provision of a public good "crowds out" private contributions". At any rate, the discussion has highlighted that: i) individuals must behave as atomistic utility maximizers, ii) the redistribution of income has to take place among current contributors of the public good, and iii) individuals must face an identical constant prices. Recent and growing literature on fiscal federalism relates with the implications of information asymmetry when local jurisdictions face different cost for the provision of public good (Cornes and Silva, 2002; Huber and Runkel, 2006). Our model shows a close relation with the work of (Huber and Runkel 2006), but with some important differences: i) we consider a pure public good while they focus their analysis on "local public good", i.e., a good which economic jurisdiction coincides with the administrative one; ii) they assume a separable utility function, whereas we don't impose any condition on the utility function; iii) they implicitly assume the same utility function for all the jurisdiction types, while we allow utilities to vary among the jurisdiction types (high or low cost); iv) they extend their analysis to the context of asymmetric information while we only focus on the perfect information case; v) they consider different transfer policies while we limit our analysis to the lump sum.

In this note, at first we provide the general rule that the Central Government has to follow in order to improve the social welfare by means of lump-sum transfer. This rule is provided in a fairly general setting, where no particular assumptions on utility and cost functions are set. The sign of the transfer, i.e., the direction from the high cost region to the low one or vice versa, is not a trivial result. Secondly we define the conditions that, if met, allow for a autonomous (i.e., without requiring any Central Government intervention) money transfer among jurisdictions when the latter behave according to the Nash rule.

The paper is organized as follows: in Section 2 the model is presented. Section 3 analyses the first best outcome under perfect information. Section 4 analyses Nash equilibriums under different price hypothesis. Finally, in Section 4 some concluding remarks are presented.

2 The model

The model assumes an economic federation consisting of two tiers of government: a central government and two local governments (we will refer to these latter as regions). Regional utility directly represents the preferences of citizenship, since the local governments aim at individualistic utility maximization. Each region provides two goods: the private good y and the public good x. The production cost for the private good y is identical among jurisdictions and set equal to 1. The cost for the public good x differs according to the jurisdiction's type. We distinguish between the low cost region's type and the high one, denoting the former by the l index and the latter by the h index. The federation comprises a low cost type region l and a high cost region h[1]. The type $i \in \{l,h\}$

[1] The model can be easily extended assuming $L>1$ ($l=1....L$) number of low cost type identical regions and $H>1$ ($h=1...H$) of high cost identical regions

region faces an expenditure cost $E^i(\theta_i, x_i)$ on x which depends and increases both on the quantity of the public good x_i provided, and on the θ_i cost parameter, assuming $\theta_h > \theta_l$. The latter characteristic is rendered explicit by the following derivatives: $E^i_x; E^i_\theta > 0$; $E^i_{xx}; E^i_{x\theta} \geq 0$ (the subscript indicates the variable with respect to which the E cost function has been derived, either at first or second order). The maximization problem that faces the region type $i \in \{l, h\}$ is given by $Max\ u^i = U^i(y_i, X)$, *where* $X = x^l + x^h$ subject to the budget constraint $R^i + \tau^i = y_i + E^i$, where R^i is the region's i income and τ^i is a lump-sum transfer (either positive or negative) set by the central government. We adopt standard assumptions for the $U(\cdot)$ function: it is increasing in y and X and strictly quasiconcave, as well as that all goods are normal. In order to maximize the utility function subject to the budget constraint, each region chooses the amount of y and x to be provided, so that[2]: $U^i_x = U^i_y E^i_x$ or equivalently $SMS^i_{x,y} = -E^i_x$. The social welfare maximization implies that the standard condition for efficiency is met: $SMS^l_{X,y^l} + SMS^h_{X,y^h} = -E^h_{x^h} = -E^l_{x^l}$

3 Social welfare and perfect information

The Central Government can transfer money (by lump sum) from one jurisdiction to the other (under the constraint to satisfy the condition for public budget balance) pursuing the goal of social welfare improvement. In the case of perfect information we assume that the Centre knows all the characteristics of the two jurisdictions as well as the type (high or low) of each jurisdiction.

In order to improve the social welfare, the central government can implement a money transfer from one region to the other, but the direction of this transfer is not trivial, in fact it might move from the high type region to the low or vice versa. The "necessary and sufficient" information required to identify the transfer sign is provided by the marginal utility on good y (U_y) and by the marginal expenditure on the public good (E_x). A first best outcome might be attained when $U^i_y = U^j_y$ and $E^i_x = E^j_x$. As a consequence, if $U^i_y > U^j_y$ or $E^i_x < E^j_x$ then i has to be subsidized while j taxed, the opposite applies in the case that $U^i_y < U^j_y$ or $E^i_x > E^j_x$. The transfer has to equalize at the margin the utility (with respect to the private good) and the expenditure (with respect to the public good) of the two regions. The intuition underlying the condition $E^i_x = E^j_x$ is straightforward: since good x is a public good, then its production has to be set in order to minimize its producing cost, given the optimal amount of public good. In other words the production has to split between the two jurisdictions so to contain as much as possible the overall cost.

4 Nash equilibrium

Assuming the Nash behaviour, the regional utility maximization requires $SMS^i_{x,y} = -E^i_x$, $i=h,l$. The implication is straightforward: when the good x is a pure public good, it is not possible to reach a first best by means of a transfer of money among the jurisdictions, even in presence of perfect information if regions are autonomous in their spending decision. In fact a first best might only be obtained by exogenously imposing the optimal expenditure levels $E^{i\,*}_x$ and y_i^*, $i=l,h$, which is tantamount to say that a Leviathan sets (and forces) the optimal values suppressing the regional autonomy and assuming jurisdictions as plants of a unique firm.

It can be shown (see appendix for details) that a money transfer between the two types of local governments would yield the following ratio in terms of utility change:

[2] The subscript indicates the derivative with respect to that variable, i.e., for instance $U_x \equiv \partial U(.)/\partial x$

$$\frac{dU^i}{dU^j}=\frac{-U_y^i\left\{1+E_x^i\left(\frac{\partial x_j}{\partial R_i}-\frac{\partial x_j}{\partial R_j}\right)\right\}}{U_y^l\left\{1+E_x^j\left(\frac{\partial x_i}{\partial R_j}-\frac{\partial x_i}{\partial R_i}\right)\right\}} \tag{1}$$

The sign of $\frac{dU^h}{dU^l}$ can turn to be greater, lower or equal to zero. The transfer sign is determined by the ratio of (1): if $\frac{dU^h}{dU^l}>0$ the transfer should move from the high cost region to the low, if $\frac{dU^h}{dU^l}<0$ the opposite applies. The transfer that follows the afore-mentioned rule allows for a social welfare improvement, even in the context of Nash behaviour.

4.1 Linear prices

In the case that the second derivative of the cost function with respect to good *x*, both for *l* and *h*, are equal to zero (i.e., $E_{xx}^l=E_{xx}^h=0$), then constant prices for the public good are implicitly assumed. (1) can then be rewritten as follows

$$\frac{dU^i}{dU^j}=\frac{U_y^i}{U_y^j}\,\frac{\dfrac{\partial\left[\dfrac{U_x^i}{U_y^i}\right]}{\partial x_i}\dfrac{\partial\left[\dfrac{U_x^i}{U_y^i}\right]}{\partial y_i}[E_x^j-E_x^i]}{\dfrac{\partial\left[\dfrac{U_x^j}{U_y^j}\right]}{\partial x_j}\dfrac{\partial\left[\dfrac{U_x^j}{U_y^j}\right]}{\partial y_j}[E_x^j-E_x^i]} \tag{2}$$

where $\frac{\partial\left[\frac{U_x^i}{U_y^i}\right]}{\partial x_i}$; $\frac{\partial\left[\frac{U_x^j}{U_y^j}\right]}{\partial x_j}$ are always negative, and $\frac{\partial\left[\frac{U_x^i}{U_y^i}\right]}{\partial y_i}$; $\frac{\partial\left[\frac{U_x^j}{U_y^j}\right]}{\partial y_j}$ are always positive (see appendix for details). It clearly emerges that the ratio $\frac{dU^i}{dU^j}$ will be always positive. This result implies that it is possible to improve the social welfare transferring money from the high cost region to the low cost one. To be noted that the term $[E_x^j-E_x^i]$ is positive when the *j* region price for the public good is greater than *i*. In that case both dU^i and dU^j are positive, otherwise both dU^i and dU^j are negative.

This statement shows a very important consequence for the equilibrium. In fact, it turns out that, in a Nash behaviour framework and without any central authority intervention, local jurisdictions autonomously proceed to transfer money each other, in particular the money transfer moves from the high cost region to the low. In other words, in the presence of a pure public good and linear prices, then the Nash behaviour approaches the social welfare goal. A money transfer allows for a utility increase both for the receiver and the donor. This result coincides with that provided by Buchholz and Konrad (1995).

Furthermore assuming both $E_{xx}^l=E_{xx}^h=0$ and $E_x^l=E_x^h$ (that makes the model to converge to the case in which regions face a identical constant prices in the production of the pure public good *x*) then a income redistribution would be ineffective. According to Warr (1983) a redistribution among jurisdictions would not affect the overall level of public good individually supplied.

Furthermore even the individual consumption of the private good would remain constant regardless to the income redistribution.

4.2 Non linear prices

If we get back to the assumption of non constant production costs (i.e., $E_x^i > 0; E_{xx}^i > 0; E_x^i \neq E_x^j$) the outcome of a autonomous money transfer among governments is still a possible scenario. Let consider the local government *i* indirect utility: $V^i(e^i, \tau, x^j, \vartheta^i) = \max U[(R^i - \tau - e^i), (B(\vartheta^i, e^i) + x^j)]$, where $B(\vartheta^i, e^i) = x^i$ is obtained by inverting the cost function $e^i = E^i(\vartheta^i, x^i)$, with $B_e(\vartheta^i, e^i) > 0; B_{ee}(\vartheta^i, e^i) \leq 0$.

By the indirect utility it is possible to derive the condition that makes government *i* to benefit from a autonomous money transfer to *j* when governments act according to the Nash rule.

Assuming for simplicity a money transfer equal to 1 ($d\tau = 1$) the condition can be written as:

$$dx^j > \frac{U_\tau^i + [U_e^i - U_B^i B_e] de^i}{U_{x^j}^i} \tag{3}$$

In the case that the condition of (3) is met, then the local jurisdiction *i* might find it profitable to autonomously proceed with money transfers to *j* in order to improve its own utility and in so doing improving also the other region utility. As well as in the case of constant prices it emerges that the Nash best behaviour moves in the direction of social welfare improvement, even though a social welfare optimum is never reachable.

5 Concluding remarks

In this paper we provide a model in order to analyse the efficient allocation of resources in a Nash behaviour context where a pure public good, that shows variable cost in its production, is involved. Never a first best outcome is attainable, even under perfect information, when a pure public good is provided by local governments that behave à la Nash. However a lump sum money transfer may yield a welfare improvement, but the transfer has to be set according to the rule provided in the paper. In the limiting case of constant and identical (among jurisdictions) price for the public good, it emerges the ineffectiveness of any redistribution policy (Warr 1983). On the other hand, assuming constant prices that however varies among jurisdictions, the outcome of Buchholz and Konrad (1995) is confirmed: the jurisdiction with the higher cost has an incentive to make unconditional transfers to the other jurisdiction and, as a consequence, it turns out that the Nash behaviour moves in the desired direction of social welfare improvement.

Furthermore when public good price is non linear then it emerges the quite unexpected result that a Nash voluntary transfer among jurisdictions (with a consequent social welfare improvement) is a possible outcome. The required condition has been highlighted in the paper and, although this condition might turn to be unusual, nonetheless it could occur allowing, in that case, for a social welfare improvement (without the need of any central authority intervention). A Nash voluntary transfer (with non linear prices) takes place whenever the local government income reduction, which in turn implies a loss in terms of utility, is more than compensated by a utility gain originated by the overall public good provision. Individuals' cross elasticities of marginal utility with respect to income and expenditure make it a possible scenario.

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Appendix

From the first order conditions we derive the best reply function for the two type of jurisdictions. Solving the simultaneous system of equations so determined, it is possible to obtain the Nash (general) equilibrium values:

$$\begin{cases} x_i = \xi^{x_i}\left(R_i, \theta_i, R_j, \theta_j\right) \\ y_i = \xi^{y_i}\left(R_i, \theta_i, R_j, \theta_j\right) \\ x_j = \xi^{x_j}\left(R_i, \theta_i, R_j, \theta_j\right) \\ y_j = \xi^{y_j}\left(R_i, \theta_i, R_j, \theta_j\right) \end{cases}$$

differentiating the system of first order conditions[3] we get:

$$\begin{cases} \left\{ \dfrac{\partial\left[\dfrac{U_x^i}{U_y^i}\right]}{\partial x_i} - E_{xx}^i \right\} dx_i + \dfrac{\partial\left[\dfrac{U_x^i}{U_y^i}\right]}{\partial x_j} dx_j + \dfrac{\partial\left[\dfrac{U_x^i}{U_y^i}\right]}{\partial y_i} dy_i = E_{x\theta}^i d\theta_i \\ \dfrac{\partial\left[\dfrac{U_x^j}{U_y^j}\right]}{\partial x_i} dx_i + \left\{ \dfrac{\partial\left[\dfrac{U_x^j}{U_y^j}\right]}{\partial x_j} - E_{xx}^j \right\} dx_j + \dfrac{\partial\left[\dfrac{U_x^j}{U_y^j}\right]}{\partial y_i} dy_j = E_{x\theta}^j d\theta_j \\ E_x^i dx_i + dy_i = dR_i - E_\theta^i d\theta_i \\ E_x^j dx_j + dy_j = dR_j - E_\theta^j d\theta_j \end{cases}$$

Defining: $A = \dfrac{\partial\left[\dfrac{U_x^i}{U_y^i}\right]}{\partial x_i} = \dfrac{\partial\left[\dfrac{U_x^i}{U_y^i}\right]}{\partial x_j} = \dfrac{U_{xx}^i U_y^i - U_x^i U_{yx}^i}{\left[U_y^i\right]^2} < 0$ & $B = \dfrac{\partial\left[\dfrac{U_x^j}{U_y^j}\right]}{\partial x_i} = \dfrac{\partial\left[\dfrac{U_x^j}{U_y^j}\right]}{\partial x_j} = \dfrac{U_{xx}^j U_y^j - U_x^j U_{yx}^j}{\left[U_y^j\right]^2} < 0$

$C_i = \dfrac{\partial\left[\dfrac{U_x^i}{U_y^i}\right]}{\partial y_i} = \dfrac{U_{xy}^i U_y^i - U_x^i U_{yy}^i}{\left[U_y^i\right]^2} > 0$ & $C_j = \dfrac{\partial\left[\dfrac{U_x^j}{U_y^j}\right]}{\partial y_j} = \dfrac{U_{xy}^j U_y^j - U_x^j U_{yy}^j}{\left[U_y^j\right]^2} > 0$

we obtain the matrix: $H = \begin{vmatrix} A - E_{xx}^i & A & C^i & 0 \\ B & B - E_{xx}^j & 0 & C^j \\ E_x^i & 0 & 1 & 0 \\ 0 & E_x^j & 0 & 1 \end{vmatrix}$

the determinant of which is: $\left|H\right| = -E_x^i C_i (B - E_{xx}^i) + (A - E_{xx}^i)(B - E_{xx}^j) - AB - E_x^j C_j (A - E_{xx}^i - C_i E_x^i) > 0$

[3] $$\begin{cases} \dfrac{U_x^i}{U_y^i} - E_x^i = 0 \\ \dfrac{U_x^j}{U_y^j} - E_x^j = 0 \\ y_i + E^i = R_i \\ y_j + E^j = R_j \end{cases}$$

$$\frac{dx_i}{dR_i} = -\frac{C_i}{|H|}(B - E^j_{xx} - C_j E^j_x) > 0 \,;\, \frac{dx_i}{dR_j} = \frac{C_j}{|H|} A < 0 \,;\, \frac{dx_j}{dR_i} = \frac{C_i}{|H|} B < 0 \,;$$

$$\frac{dx_j}{dR_j} = -\frac{C_j}{|H|}(A - E^i_{xx} - C_i E^i_x) > 0 \,;\, \frac{dy_i}{dR_i} = -\frac{AE^j_{xx} + E^i_{xx}(B - E^j_{xx})}{|H|} - \frac{C_j E^j_x (A - E^i_{xx})}{|H|} > 0 \,;\, \frac{dy_i}{dR_j} = -\frac{C_j}{|H|} AE^i_x > 0$$

Let's first assume that a income variation in region *i* occurs.

a) $dR_i \neq 0$ & $dR_j = 0$

$$dU^i = \left[U^i_x\left(\frac{dx_i}{dR_i} + \frac{dx_j}{dR_i}\right) + U^i_y \frac{dy_i}{dR_i}\right] dR_i = \left[U^i_y E^i_x\left(\frac{dx_i}{dR_i} + \frac{dx_j}{dR_i}\right) + U^i_y \frac{dy_i}{dR_i}\right] dR_i = U^i_y \left[E^i_x\left(\frac{dx_i}{dR_i} + \frac{dx_j}{dR_i}\right) + \frac{dy_i}{dR_i}\right] dR_i$$

$$dU^i = U^i_y \left[\left(E^i_x \frac{dx_i}{dR_i} + \frac{dy_i}{dR_i}\right) + E^i_x \frac{dx_j}{dR_i}\right] dR_i \;\Rightarrow$$

$$E^i_x \frac{dx_i}{dR_i} + \frac{dy_i}{dR_i} = \left[-E^i_x \frac{C_i}{|H|}(B - E^j_{xx} - C_j E^j_x)\right] + \left[-\frac{AE^j_{xx} + E^i_{xx}(B - E^j_{xx})}{|H|} - \frac{C_j E^j_x (A - E^i_{xx})}{|H|}\right] =$$

$$= \frac{-E^i_x C_i (B - E^j_{xx} - C_j E^j_x) - AE^j_{xx} - E^i_{xx}(B - E^j_{xx}) - C_j E^j_{xx}(A - E^i_{xx})}{-E^i_x C_i (B - E^j_{xx} - C_j E^j_x) - AE^j_{xx} - E^i_{xx}(B - E^j_{xx}) - C_j E^j_{xx}(A - E^i_{xx})} = 1$$

$$\frac{dU^i}{dR_i} = U^i_y \left(1 + E^i_x \frac{dx_j}{dR_i}\right) >,=< 0 \quad se \quad 1 >,=,< -E^i_x \frac{dx_j}{dR_i}$$

But using the following information:

$\frac{dX}{dR_i} = \frac{dx_i}{dR_i} + \frac{dx_j}{dR_i} = -\frac{C_i}{|H|}(B - E^j_{xx} - C_j E^j_x) + \frac{C_i}{|H|} B = \frac{C_i}{|H|}(E^j_{xx} + C_j E^j_x) > 0$, and also knowing that $\frac{dy_i}{dR_i} > 0$, then $\frac{dU^i}{dR_i} > 0$, and therefore $1 > -E^i_x \frac{dx_j}{dR_i}$. Which in turn implies that:

$$\frac{dU^i}{dR_i} = U^i_y \left(1 + E^i_x \frac{dx_j}{dR_i}\right) > 0$$

Assuming now that only region *j* faces a income variation:

b) $dR_i = 0$ & $dR_j \neq 0$

$$dU^i = U^i_y \left[E^i_x\left(\frac{dx_i}{dR_j} + \frac{dx_j}{dR_j}\right) + \frac{dy_i}{dR_j}\right] dR_j = U^i_y \left[E^i_x \frac{dx_i}{dR_j} + \frac{dy_i}{dR_j}\right] dR_j + U^i_y E^i_x \frac{dx_j}{dR_j} dR_j$$

$$E^i_x \frac{dx_i}{dR_j} + \frac{dy_i}{dR_j} = \left[E^i_x \frac{C_j}{|H|} B\right]\left[-\frac{C_j}{|H|} BE^i_x\right] = 0$$

$$\frac{dU^i}{dR_j} = U^i_y E^i_x \frac{dx_j}{dR_j} > 0$$

Similarly to the previous case, let's define the region's *j* utility variation:

$$\frac{dU^j}{dR_j} = U^j_y \left(1 + E^j_x \frac{dx_i}{dR_j}\right) > 0 \quad \& \; 1 > -E^j_x \frac{dx_i}{dR_j}$$

whereas assuming $dR_i \neq 0$ & $dR_j = 0$

$$\frac{dU^j}{dR_i} = U^j_y E^j_x \frac{dx_i}{dR_i} > 0$$

Considering the utility variation for both regions when a money transfer from *i* to *j* occurs:

We assume that region i is characterized by higher cost (with respect to region j) in providing the public good; i.e., $E_x^i > E_x^j$; $E_{xx}^i > E_{xx}^j$. When a transfer between the two regions occurs, it is tantamount to say that both regions face a income variation, equal in absolute value, but different in sign: $dR=dR_i=-dR_j>0$.
Let's first assume a income variation in region i:

The utility of j varies accordingly: $dU^j = \left[U_y^j (1+E_x^j \frac{dx_i}{dR_j}) \right] dR_j + \left[U_y^j E_x^j \frac{dx_i}{dR_i} \right] dR_i$.

Thus: $dU^j = U_y^j \left[1+E_x^j \frac{dx_i}{dR_j} - E_x^j \frac{dx_i}{dR_i} \right] dR \Rightarrow dU^j = U_y^j \left[1+E_x^j \left(\frac{dx_i}{dR_j} - \frac{dx_i}{dR_i} \right) \right] dR$

We know that $\frac{dx_i}{dR_j} - \frac{dx_i}{dR_i} < 0$. Thus a money transfer from i to j will cause a utility increase to the j region if: $U_y^j + U_y^j E_x^j \left(\frac{dx_i}{dR_j} - \frac{dx_i}{dR_i} \right) > 0 \Rightarrow 1 > E_x^j \left(\frac{dx_i}{dR_i} - \frac{dx_i}{dR_j} \right) > 0$, or equivalently if:

$$E_x^j < \frac{1}{\frac{dx_i}{dR_i} - \frac{dx_i}{dR_j}}$$

A money transfer from i to j is suitable to produce a utility increase in j when the utility increase in the latter (region j) originated by the larger consumption of the private good y is greater than the utility loss consequent to the diminished contribution in the public good X by region i

Similarly $dU^i = -\left[U_y^i (1+E_x^i \frac{dx_j}{dR_i}) \right] dR + \left[U_y^i E_x^i \frac{dx_j}{dR_j} \right] dR$

$$dU^i = -U_y^i \left[1+E_x^i \left(\frac{dx_j}{dR_i} - \frac{dx_j}{dR_j} \right) \right] dR,$$

but considering that $\frac{dx_j}{dR_i} - \frac{dx_j}{dR_j} < 0$,

we may conclude that a money transfer from i to j produces a utility increase for region i when:

$1+E_x^i \left(\frac{dx_j}{dR_i} - \frac{dx_j}{dR_j} \right) < 0$, that is equivalent to say: $1 < E_x^i \left(\frac{dx_j}{dR_j} - \frac{dx_j}{dR_i} \right)$

$$E_x^i > \frac{1}{\frac{dx_j}{dR_j} - \frac{dx_j}{dR_i}}.$$

Thus, to sum up, a money transfer from a region to the other determines a ratio of utility variation equal to: $\frac{dU^j}{dU^i} = -\frac{U_y^i}{U_y^b} \frac{1+E_x^j \left(\frac{dx_i}{dR_j} - \frac{dx_i}{dR_i} \right)}{1+E_x^a \left(\frac{dx_j}{dR_i} - \frac{dx_j}{dR_j} \right)}$