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Long-run strong-exogeneity

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Abstract

This note supplements the paper by Pradel and Rault (2003) "Exogeneity in VAR-ECM models with purely exogenous long-run paths", Oxford Bulletin of Economics and Statistics. In particuliar, we propose a condition to distinguish between cointegration amongst "endogenous" and "exogenous" variables and also between cointegrating vectors appearing in the equations of the "endogenous" and "exogenous" variables, i.e in the conditional and marginal models. This condition that we call "long-run strong-exogeneity" has a practical appealing aspect since it permits valid long-run forecasts from the conditional model alone.

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Consider a Gaussian VAR of order p in error-correction form for an n-dimensional I(1) vector time series $\{X_t\}$:

$$\Delta X_t = \sum_{i=1}^{p-1} \Gamma_i \Delta X_{t-i} + \alpha\beta^{'} X_{t-1} + \varepsilon_t, t = 1, ..., T, \tag{1}$$

with fixed initial values of X_{-p+1},..,X_0 and where ε_t is a n-dimensional homoskedastic Gaussian innovation process with a zero mean and a positive definite covariance matrix Σ. Furthermore, Γ_i, α, β are, respectively $n \times n$, $n \times r$, $n \times r$, $0 < r < n$ matrices such that $\Pi = \alpha\beta^{'}$;and p is a constant integer. The columns of β span the space of cointegrating vectors, and the elements of α are the corresponding adjustment coefficients or loading factors. For notation convenience, no deterministic components are included in the models.

We make the following (conventional) two assumptions: (i) $\left|(I_n - \sum_{i=1}^{p-1} \Gamma_i z^i)(1-z) - \alpha\beta^{'} z\right| = 0$ which implies either $|z| > 1$ or $z = 1$, and (ii) the matrix $\alpha'_\perp(I_n - \sum_{i=1}^{p-1} \Gamma_i)\beta_\perp$ is invertible, where $\beta_\perp$ and $\alpha_\perp$ are both full rank $n \times n - r$ matrices satisfying $\alpha'_\perp \alpha = \beta^{'}_\perp \beta = 0$, which rules out the possibility that one or more elements of X_t are $I(2)$. Assumptions (i) and (ii) imply (see Johansen, 1995) that the process X_t is cointegrated of order (1,1).

Consider now the partition of the n dimensional cointegrated vector time series $X_t = (Y_t', Z_t')'$ generated by equation (1), where Y_t and Z_t are distinct g $\times$ 1 and k $\times$ 1 subvectors of variables that we call "endogenous" and "exogenous", with $g + k = n$, as well as the following theorem proved in Pradel and Rault (2003) (cf theorem 2, p 636):

Theorem 1 *Let* $\Pi = \alpha\beta^{'}$ *be a* $n \times n$ *reduced rank matrix of rank* r $(0 < r < n)$ *and consider the reparametrisation*

$$\beta = [\beta_1 \ \beta_2] = \begin{bmatrix} \beta_{YY} & 0 \\ \beta_{ZY} & \beta_{ZZ} \end{bmatrix}, \alpha = [\alpha_1 \ \alpha_2] = \begin{bmatrix} \alpha_{YY} & \alpha_{YZ} \\ \alpha_{ZY} & \alpha_{ZZ} \end{bmatrix}$$

given in theorem 1 (cf Rault and Pradel, 2003, p 634) , where β_{YY}, α_{YY}, β_{ZY}, α_{ZY}, α_{YZ},β_{ZZ}, α_{ZZ} *are respectively* $g \times r_1$, $g \times r_1$, $k \times r_1$, $k \times r_1$, $g \times r\text{-}r_1$, $k \times r\text{-}r_1$, $k \times r\text{-}r_1$ *sub-matrices, with* $rank\,(\beta_{YY}) = r_1 > 0$ *and* $rank\,(\beta_{ZZ}) = r - r_1 > 0$[1]. *Then :*

(i) there exists an integer r_2 *so that the* α *et* β *matrices can always be reparametrised as follows:*

$$\begin{aligned} \alpha &= [\alpha_1 \, \alpha_{21} \, \alpha_{22}] = \begin{bmatrix} \alpha_{YY} & \alpha_{YZ_1} & \alpha_{YZ_2} \\ \alpha_{ZY} & 0 & \alpha_{ZZ_2} \end{bmatrix} \\ \beta &= [\beta_1 \, \beta_{21} \, \beta_{22}] = \begin{bmatrix} \beta_{YY} & 0 & 0 \\ \beta_{ZY} & \beta_{ZZ_1} & \beta_{ZZ_2} \end{bmatrix} \end{aligned}$$

, where α_{YZ_1}, β_{ZZ_1}, α_{YZ_2}, α_{ZZ_2}, β_{ZZ_2} *are respectively* $g \times r^*$, $k \times r^*$, $g \times r_2$, $k \times r_2$, $k \times r_2$ *sub-matrices, with* $r_1 + r_2 + r^* = r$ *and* $rank\,(\alpha_{ZZ_2}) = r_2 \geq 0$.

(ii) if in addition $\alpha_{ZY} = 0$ *(or* $r_1 = 0$*), then* r_2 *is uniquely defined and is invariant to the chosen reparametrisation. It is such as*[2]

$$\max\,(0, r - r_1 - g) \leq r_2 \leq min(g, k, r).$$

Given theorem 1 (i), equation (1) can be rewritten as a conditional model for Y_t given Z_t and a marginal model for Z_t, that is :

[1] We assume that β_1 and β_2 each contain at least one cointegrating vector to exclude the case where β_1=β, which entails that β_2 is a null set.

[2] Remind that $rank\,(\alpha) = r$.

$$\begin{cases} \text{conditional model} \\ \Delta Y_t = \sum_{i=1}^{p-1} \Gamma^{+}_{YY,i} \Delta Y_{t-i} + \sum_{i=0}^{p-1} \Gamma^{+}_{YZ,i} \Delta Z_{t-i} + \alpha^{+}_{YY} \beta^{'}_{1} X_{t-1} + \left(\alpha^{+}_{YZ_1} \beta^{'}_{ZZ_1} + \alpha^{+}_{YZ_2} \beta^{'}_{ZZ_2} \right) Z_{t-1} + \eta_{Y,t} \\ \text{marginal model} \\ \Delta Z_t = \sum_{i=1}^{p-1} \Gamma_{ZY,i} \Delta Y_{t-i} + \sum_{i=1}^{p-1} \Gamma_{ZZ,i} \Delta Z_{t-i} + \alpha_{ZY} \beta^{'}_{1} X_{t-1} + \alpha_{ZZ_2} \beta^{'}_{ZZ_2} Z_{t-1} + \varepsilon_{Z,t} \end{cases} \tag{2}$$

$$\text{with} \begin{cases} \Gamma^{+}_{YY}(L) = \Gamma_{YY}(L) - \Sigma_{YZ} \Sigma^{-1}_{ZZ} \Gamma_{ZY}(L) = I_g - \sum_{i=1}^{p-1} \Gamma^{+}_{YY,i} L^i \\ \Gamma^{+}_{YZ}(L) = \Gamma_{YZ}(L) - \Sigma_{YZ} \Sigma^{-1}_{ZZ} \Gamma_{ZZ}(L) = - \sum_{i=0}^{p-1} \Gamma^{+}_{YZ,i} L^i \\ \alpha^{+}_{YY} = \alpha_{YY} - \Sigma_{YZ} \Sigma^{-1}_{ZZ} \alpha_{ZY} \\ \alpha^{+}_{YZ_1} = \alpha_{YZ_1} \\ \alpha^{+}_{YZ_2} = \alpha_{YZ_2} - \Sigma_{YZ} \Sigma^{-1}_{ZZ} \alpha_{ZZ_2} \\ \eta_{Yt} = \varepsilon_{Yt} - \Sigma_{YZ} \Sigma^{-1}_{ZZ} \varepsilon_{Zt} \\ \Sigma^{+}_{YY} = \Sigma_{YY} - \Sigma_{YZ} \Sigma^{-1}_{ZZ} \Sigma_{ZY} \end{cases}$$

where L denotes the lag operator

and $\begin{pmatrix} \eta_{Yt} \\ \varepsilon_{Zt} \end{pmatrix} \sim N \left[\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} \Sigma^{+}_{YY} & 0 \\ 0 & \Sigma_{ZZ} \end{pmatrix} \right]$

with the partitioning of the matrices Γ_i, α and β being conformable to that of X_t.

In this framework it is now possible to draw up a condition to distinguish between cointegration amongst "endogenous" and "exogenous" variables and also between cointegrating vectors appearing in the equations of the "endogenous" and "exogenous" variables, i.e. in the conditional and marginal models. In this case the standard partition of "endogenous" and "exogenous" holds in the long run even when the variables are jointly cointegrated. This condition may be seen as a new concept of exogeneity, that we call "long-run strong-exogeneity". Such separate cointegration can arise for instance when the Y_t variables define a market relationship and some of the Z_t variable effects relate to spillover from micro markets or in an investment equation when the "exogenous" variable cointegration is due to a dependence amongst asset or between asset prices and inflation. We can now state the following definition :

Definition 1 *: **Long-run strong exogeneity of** Z_t*

Z_t is said to be strongly exogenous in the long-run for the parameters of interest if and only if :

(i) Z_t is weakly exogenous for the parameters of interest[3]*,*

(ii) Y doesn't cause Z in the long-run in Granger sense (1969), i.e. Π_{ZY}*=0.*

Long-run strong-exogeneity is distinct from weak and strong-exogeneity but is most closely akin to strong-exogeneity because it includes weak-exogeneity and long-run non-causality. Besides it only emerges in a VAR-ECM model since it requires the existence of two different chanels of causality, a short-run and a long-run causality[4]. In a VAR model it is similar to strong-exogeneity.

Proposition 1 ***: Necessary and sufficient condition for long-run strong exogeneity***

Suppose that the parameters of interest are those of the conditional model (cf. equation 2), i.e. Ψ $=(\Gamma^{+}_{YY,i},\ i = 1,...,p-1;\ \Gamma^{+}_{YZ,i}, i = 0,...,p-1;\ \alpha^{+}_{YY};\beta^{'}_{1};\alpha^{+}_{YZ_1};\beta^{'}_{ZZ_1})$. *If* $r_2 < k$ *then Z_t is strongly exogenous in the long-run for Ψ if and only if* $\left\{ \begin{array}{l} \alpha_{ZY} = 0 \\ \alpha^{+}_{YZ_2} = 0 \end{array} \right.$ *in the canonical representation given by theorem 1.*

The proof follows the same line of arguments as those presented in Rault and Pradel (2003) and is omitted here to save space. Notice that when α and β are given by proposition 1

(i)

$$\Pi = \alpha\beta^{'} = \left[\begin{array}{c} \alpha^{+}_{YY}\beta^{'}_{1} \\ 0 \end{array} \right] + \left[\begin{array}{ccc} 0 & \alpha^{+}_{YZ_1}\beta^{'}_{ZZ_1} & 0 \\ 0 & 0 & \alpha_{ZZ_2}\beta^{'}_{ZZ_2} \end{array} \right],$$

[3]Let's remember that Engle *et al* (1983) define a vector of Z_t variables to be weakly-exogenous for the parameters of interest, if (i) the parameters of interest only depend on those of the conditional model, (ii) the parameters of the conditional and marginal models are variation free, i.e. there exists a sequential cut of the two parameters spaces (cf. Florens and Mouchart, 1980).

[4]See Rault (2000) for futher details.

(ii) there always exists bases of the corresponding orthogonal spaces which can be written as :

$$\alpha_\perp = \begin{bmatrix} 0 & 0 & \alpha_{YZ_{2\perp}} \\ \alpha_{ZY\perp} & \alpha_{ZZ_{1\perp}} & \alpha_{ZZ_{2\perp}} \end{bmatrix}, \qquad \beta_\perp = \begin{bmatrix} \beta_{YY\perp} & 0 & \beta_{YY2\perp} \\ \beta_{ZY\perp} & \beta_{ZZ_{1\perp}} & 0 \end{bmatrix}$$

where $\alpha_{ZY\perp}$, $\beta_{YY\perp}$, $\beta_{ZY\perp}$, $\alpha_{ZZ_{1\perp}}$, $\beta_{ZZ_{1\perp}}$, $\alpha_{YZ_{2\perp}}$, $\beta_{YY2\perp}$, $\alpha_{ZZ_{2\perp}}$ are respectively k $\times$ k-r$_2$-r*, g $\times$ k-r$_2$-r*,k $\times$ k-r$_2$-r*, k $\times$ r*, k $\times$ r*, g $\times$ g-r$_1$-r*, g $\times$ g-r$_1$-r*, k $\times$ g-r$_1$-r* sub-matrices, such that the matrix $\alpha'_\perp(I_n - \sum_{i=1}^{p-1}\Gamma_i)\beta_\perp$ has always full rank, which shows that proposition 1 does not involve I(2) variables.

Remark 1 *: Comment (ii) highlights a possible problem with Proposition 2 of Pradel and Rault (2003) which may be solved by the present Proposition 1. Pradel and Rault define strong exogeneity by the parameter restrictions:*

$$\alpha = \begin{bmatrix} \alpha_{YY} & \alpha_{YZ_1} & 0 \\ 0 & 0 & \alpha_{ZZ_2} \end{bmatrix}, \beta = \begin{bmatrix} \beta_{YY} & 0 & 0 \\ \beta_{ZY} & \beta_{ZZ_1} & \beta_{ZZ_2} \end{bmatrix}, \Gamma = \begin{bmatrix} \Gamma_{YY,i} & \Gamma_{YZ,i} \\ 0 & \Gamma_{ZZ,i} \end{bmatrix}$$

However, in such system $\beta'_{ZZ_1}Z_t$ can never become stationary because it only effects ΔY_t and ΔY_t does not effect ΔZ_{t+j}. Hence there is no correction in Z_t towards the equilibrium $\beta'_{ZZ_1}Z_t = 0$. This will lead to a deficient rank of $\alpha'_\perp(I_n - \sum_{i=1}^{p-1}\Gamma_i)\beta_\perp$. As an example, take $n = 3, g = 2, k = 1, r = 2, r_1 = 1, r_2 = 0, r^* = 1,$ and

$$\alpha = \begin{bmatrix} \alpha_{11} & \alpha_{12} \\ \alpha_{21} & \alpha_{22} \\ 0 & 0 \end{bmatrix}, \beta = \begin{bmatrix} 1 & 0 \\ \beta_{21} & 0 \\ \beta_{31} & 1 \end{bmatrix}, \alpha_\perp = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \beta_\perp = \begin{bmatrix} -\beta_{21} \\ 1 \\ 0 \end{bmatrix}.$$

In this case $\Gamma_{31,i} = \Gamma_{32,i} = 0$ will imply $\alpha'_\perp(I_n - \sum_{i=1}^{p-1}\Gamma_i)\beta_\perp = 0$, so the system is not I (1) cointegrated. The same will happen in more general systems. Therefore the strong exogeneity conditions of Proposition 2 of Pradel and Rault cannot coexist in an I(1) model, unless $r^* = 0$. By loosening the restrictions on $\Gamma_{ZY,i}$ as in Proposition 1 of the present paper, this problem is solved.

Long-run strong-exogeneity is the condition that the long-run cointegrating relations between a set of Y_t and Z_t variables are block triangular. Under this condition a subset of cointegrating relations only including Z_t variables may feed back onto all variables but cointegrating relations between Y_t and Z_t variables do not feed back onto the subset. Thus, for long-run purposes the subset of variables may be forecast without considering long-run relations involving the remaining variables. Moreover, the Data Generating Processes of the conditional and marginal models operate a partial separation and valid long-run forcasts of Y_t can be carried out from the conditional model alone given forecasts of Z_t. Long-run strong-exogeneity is therefore a useful concept which helps to reduce the complexity of large systems, reduce computional expense and permits simpler modelling strategies. Besides, it can easily be investigated in applied studies using Johansen and Juselius's (1990) procedures for testing hypotheses about the cointegrating vectors and the weighting coefficients since certain zero restrictions both on the α and β matrices corresponds to long-run strong-exogeneity.

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