Economics Bulletin

Volume 31, Issue 1

Manipulation of the Borda rule by introduction of a similar candidate

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Abstract

In an election contest, a losing candidate a can manipulate the election outcome in his favor by introducing a weak similar candidate WSC in the choice set, the WSC b being defined as an alternative which is ranked immediatly below a in the individual preferences. We characterize the voting situations where this manipulation is efficient for the Borda rule and express its vulnerability for a 3 alternative election.

I am grateful to Dominique Lepelley and Vincent Merlin for their helpful comments.
Citation: Jérôme Serais, (2011) "Manipulation of the Borda rule by introduction of a similar candidate", *Economics Bulletin*, Vol. 31 no.1 pp. 749-758.
Submitted: Feb 04 2011. **Published:** March 02, 2011.

1 Introduction

From the theorem of Gibbard-Satterthwaite (1973, 1975) asserting that all reasonable voting procedures are sensitive to manipulation by strategic voters, an important literature has been developed on the possible strategies of voters and their efficiencies. Recently, a different way of acting upon the election result has arisen. By changing the set of alternative to choose rather than the preferences, the final outcome might be radically different and favour the manipulators. Saari proved (1989, 1990) that adding or removing a candidate may affect seriously the ranking of scoring rules. Dutta et al. (2001) asserted that all reasonable voting procedures are sensitive to strategic candidacy, *i.e.*, a candidate can alter the election outcome by deciding whether or not to enter the set of candidates.

In this paper, we will focus on a specific way of changing this set suggested by Dummett (1998) that he called *agenda manipulation.* A losing candidate may try to manipulate the election outcome by promoting the candidacy of a *Weak Similar Candidate (WSC)*, *i.e.*, another candidate who is considered almost similar to him by all the voters [1]. Dummett noticed that the Borda rule may suffer from this manipulation and gave some examples such as in Example 1.

The Borda rule (see Borda 1781) belongs to the class of point ranking rules where points are given to each candidate according to his rank in the preference of the voters. When n voters have to choose among m alternatives, each one assigns $m - r$ points to the candidate with the rank r in his individual preference. The Borda score of a candidate is the amount of points that he collects and the chosen candidate is the one with the highest Borda score. In Example 1, eleven voters have to choose among four alternatives a, b, c, and d. Each column represents a preference ordering, with the number of individuals with this specific preference, and the last column represents the number of points that a voter gives to a candidate according to his rank.

Example 1

2	3	2	1	2	1	*points*
b	c	c	a	b	b	3
d	d	a	b	a	a	2
a	a	d	d	d	c	1
c	b	b	c	c	d	0

In this example, a is chosen with a Borda score of 18 (17 for b, 16 for c and 15 for d). Dummett assumes that before the vote is taken, a fifth alternative,

[1] Dummett called this new candidate a *clone* but we prefer to define it as *weak similar* because a *clone* could be considered as equivalent to his genuine candidate. For different definitions of a clone, see Laslier (1999), Tideman (1987) and Zavist and Tideman (1989).

e, is introduced by d, whom every voter ranks immediately below d. The chosen candidate among this new choice set is now d with a Borda score of 26 (24 for a, 23 for b, 22 for c, and 15 for e).
As the reader can notice, this manipulation is sufficiently powerful to make d chosen while he had the lowest Borda score with the initial voting situation. In this example, the manipulation can radically change the outcome in favour of the manipulator and it is natural to raise the question of the theoretical likelihood of this manipulation. The procedure suggested by Gerhlein and Fishburn (1976) to obtain analytical representation of the probability of an event is of great interest for our purpose. Their method being difficult to apply for this type of manipulation, we use the algorithm provided by Huang and Chua (2000)[2] derived from the method of Gehrlein and Fishburn.
Basic notions and assumptions are introduced in Section 2. We characterize the voting situations where the manipulation by introduction of a WSC can be successful, present the method of Huang and Chua and give the vulnerability of the Borda rule to this manipulation in a three-candidate election in Section 3. We discuss results in Section 4.

2 Notations

Let X be the set of possible alternatives and A^3 a finite subset of X, with $|A| = m$. The set of individuals who choose among the candidates is $I = \{1, \ldots, i, \ldots, n\}$ with $|I| = n$. We assume that the individuals are able to rank all the alternatives without ties in X. The preferences on a finite subset A are the restriction of P_i on A. The six possible preference orderings over A will be numbered as follows when $A = \{a, b, c\}$:

Table 1

n_1	n_2	n_3	n_4	n_5	n_6
a	a	c	c	b	b
b	c	a	b	c	a
c	b	b	a	a	c

A *voting situation* is a vector $s = (n_1, n_2, n_3, n_4, n_5, n_6)$, with n_t the number of type t voters and $\sum_{t=1}^{6} n_t = n$, that gives the distribution of the n voters over the six possible preference types. S^n is the set of all possible voting situations. A *social choice function* (SCF) $g : \cup_{i=1}^{n} S^n \rightarrow A$, assigns to each voting situation a nonempty subset of A, $g(A)$. We shall assume throughout the paper that the social decision for the context A only depends upon the

[2] See also Gehrlein (2002)

[3] The cardinality of A will vary when an attempt of manipulation by introduction of a WSC takes place.

restriction of the preferences in the profile on A[4]. N_{xy} is the number of voters who prefer x to y in a voting situation s. In case of a tie, we use the lexicographic order to choose the winner.
The definition of a WSC we give, expresses the idea of a candidate creating another candidate always immediately ranked after him in the individual preferences.

Definition 1 *A candidate y is a Weak Similar Candidate of x for a voting situation s if and only if:*
$\forall z \in X \setminus \{x, y\},\ \forall i \in I,\ xP_iz \Longleftrightarrow yP_iz$ *and* $\forall i \in I,\ xP_iy$

Given a voting situation s, we say that the Borda rule is sensitive to manipulation by introduction of a WSC at s if the outcome of a vote is better for a candidate when his WSC is introduced in the context. The vulnerability of the Borda rule will be the number of voting situations where this rule is vulnerable to this manipulation at s when compared with the number of all possible voting situations.

3 Vulnerability to Dummett's manipulation

We characterize the voting situations at which the Borda rule is sensitive to this manipulation. Lemma 1 describes the case where only one losing candidate manipulate while the other losing candidate doesn't react.

Lemma 1 *Suppose $m = 3$ and consider a voting situation $s = (n_1, n_2, n_3, n_4, n_5, n_6)$. The Borda rule is sensitive to single manipulation by introduction of a WSC at s if and only if:*
$S^{ab}_{B,s} \geq 0$ *and* $S^{ac}_{B,s} \geq 0$ *and*
$\Big[(N_{ba} > S^{ab}_{B,s}$ *and* $N_{bc} \geq S^{cb}_{B,s})$ *or* $(N_{ca} > S^{ac}_{B,s}$ *and* $N_{cb} > S^{bc}_{B,s})\Big]$
$S^{ba}_{B,s} > 0$ *and* $S^{bc}_{B,s} \geq 0$ *and*
$\Big[(N_{ab} \geq S^{ba}_{B,s}$ *and* $N_{ac} \geq S^{ca}_{B,s})$ *or* $(N_{cb} > S^{bc}_{B,s}$ *and* $N_{ca} > S^{ac}_{B,s})\Big]$
$S^{ca}_{B,s} > 0$ *and* $S^{cb}_{B,s} > 0$ *and*
$\Big[(N_{ac} \geq S^{ca}_{B,s}$ *and* $N_{ab} \geq S^{ba}_{B,s})$ *or* $(N_{bc} \geq S^{cb}_{B,s}$ *and* $N_{ba} > S^{ab}_{B,s})\Big]$

Proof.
• We assume that the Borda rule is sensitive to single manipulation by introduction of a WSC and a chosen initially: $S^{ab}_{B,s} \geq 0$ and $S^{ac}_{B,s} \geq 0$.
The candidate b creates d. We obtain the following voting situation s':

[4]This is not the condition of Independance of Irrelevant Alternatives of Arrow(1963): in a context A, the social preference between x and y may depend upon all the preferences on A, not only on the pair $\{x, y\}$. But the preference on $\{x, y\}$ is not influenced by z, which is not in the menu A.

n_1	n_2	n_3	n_4	n_5	n_6	points
a	a	c	c	b	b	3
b	c	a	b	d	d	2
d	b	b	d	c	a	1
c	d	d	a	a	c	0

The new Borda scores are :
$S^a_{B,s'} = S^a_{B,s} + N_{ab}$, $S^b_{B,s'} = S^b_{B,s} + n$, $S^c_{B,s'} = S^c_{B,s} + N_{cb}$ and $S^d_{B,s'} = S^b_{B,s}$
b is chosen if he beats the other candidates. As d is a WSC of b, d is always beaten by b.
b beats a if the difference between their new score is strictly positive which is equivalent to write $N_{ba} > S^{ab}_{B,s}$ and b beats c if the difference between their new score is positive which is equivalent to write $N_{bc} \geq S^{cb}_{B,s}$.
• We assume $S^{ab}_{B,s} \geq 0$, $S^{ac}_{B,s} \geq 0$, $N_{ba} > S^{ab}_{B,s}$ *and* $N_{bc} \geq S^{cb}_{B,s}$.
$S^{ab}_{B,s} \geq 0$ *and* $S^{ac}_{B,s} \geq 0$ implies a chosen.
$N_{ba} > S^{ab}_{B,s} \iff S^b_{B,s} + n > S^a_{B,s} + N_{ab}$
$\iff 3(n_5 + n_6) + 2(n_1 + n_4) + n_2 + n_3 > 3(n_1 + n_2) + 2n_3 + n_6$ (1)
$N_{bc} \geq S^{cb}_{B,s} \iff S^b_{B,s} + n \geq S^c_{B,s} + N_{cb}$
$\iff 3(n_5 + n_6) + 2(n_1 + n_4) + n_2 + n_3 \geq 3(n_3 + n_4) + 2n_2 + n_5$ (2).
The conditions (1) and (2) correspond to the fact of having yet 6 preference types and 4 candidates as Borda gives now 3 points for a top candidate. One property of the Borda rule is that each voter of type n_t gives $m(m-1)$ points to the candidates. As we know that there is 4 candidates and the number of points given by each voter of type n_t for a, b and c we find that the new candidate d is always ranked after b in the individual preferences. We conclude that d is a WSC of b. From the conditions (1) and (2) we know that b is chosen: the Borda rule is sensitive to single manipulation by introduction of a WSC. The other cases are similar and we omitt them. *Q.E.D.*

Lemma 2 describes the situations where the two losing candidates create simultaneously a WSC. They act upon the choice set simultaneously but with a different aim. Each losing candidate introduces a WSC of him in order to be chosen.

Lemma 2 *Suppose* $m = 3$ *and consider a voting situation* $s = (n_1, n_2, n_3, n_4, n_5, n_6)$. *The Borda rule is sensitive to simultaneous manipulation by introduction of a WSC at s if and only if*
$S^{ab}_{B,s} \geq 0$ *and* $S^{ac}_{B,s} \geq 0$ *and* $(S^b_{B,s} > S^{ab}_{B,s} + N_{ac}$ *and* $S^{bc}_{B,s} \geq N_{cb} - N_{bc})$
$S^{ab}_{B,s} \geq 0$ *and* $S^{ac}_{B,s} \geq 0$ *and* $(S^c_{B,s} > S^{ac}_{B,s} + N_{ab}$ *and* $S^{cb}_{B,s} > N_{bc} - N_{cb})$
$S^{ba}_{B,s} > 0$ *and* $S^{bc}_{B,s} \geq 0$ *and* $(S^a_{B,s} \geq S^{ba}_{B,s} + N_{bc}$ *and* $S^{ac}_{B,s} \geq N_{ca} - N_{ac})$
$S^{ba}_{B,s} > 0$ *and* $S^{bc}_{B,s} \geq 0$ *and* $(S^c_{B,s} > S^{bc}_{B,s} + N_{ba}$ *and* $S^{ca}_{B,s} > N_{ac} - N_{ca})$
$S^{ca}_{B,s} > 0$ *and* $S^{cb}_{B,s} > 0$ *and* $(S^a_{B,s} \geq S^{ca}_{B,s} + N_{cb}$ *and* $S^{ab}_{B,s} \geq N_{ba} - N_{ab})$
$S^{ca}_{B,s} > 0$ *and* $S^{cb}_{B,s} > 0$ *and* $(S^b_{B,s} \geq S^{cb}_{B,s} + N_{ca}$ *and* $S^{ba}_{B,s} > N_{ab} - N_{ba})$

Proof of Lemma 2

• We assume that the Borda rule is sensitive to simultaneous manipulation by introduction of a WSC and a chosen initially. We have $S^{ab}_{B,s} \geq 0$ and $S^{ac}_{B,s} \geq 0$

We assume that b is chosen after the partisans of b and c have acted upon the choice set by creating WSC d and e. We have the following profile s'

n_1	n_2	n_3	n_4	n_5	n_6	points
a	a	c	c	b	b	4
b	c	e	e	d	d	3
d	e	a	b	c	a	2
c	b	b	d	e	c	1
e	d	d	a	a	e	0

The new Borda scores are :

$S^{a}_{B,s'} = S^{a}_{B,s} + N_{ab} + N_{ac}$, $S^{b}_{B,s'} = S^{b}_{B,s} + n + N_{bc}$, $S^{c}_{B,s'} = S^{c}_{B,s} + n + N_{cb}$, $S^{d}_{B,s'} = S^{b}_{B,s} + N_{bc}$ and $S^{e}_{B,s'} = S^{c}_{B,s} + N_{cb}$.

b is chosen if he beats the other candidates. As d is the WSC of b, d is always beaten by b:

b beats a if the difference between their new score is strictly positive: $S^{ba}_{B,s'} > 0$, which is equivalent to write $S^{b}_{B,s} > S^{ab}_{B,s} + N_{ac}$.

b beats c if the difference between their new score is positive: $S^{bc}_{B,s'} \geq 0$, which is equivalent to write $S^{bc}_{B,s} \geq N_{cb} - N_{bc}$

e is the WSC of c so we have always $S^{ce}_{B,s'} \geq 0$, as $S^{bc}_{B,s'} \geq 0$ it implies $S^{be}_{B,s'} \geq 0$.

• We assume $S^{ab}_{B,s} \geq 0$, $S^{ac}_{B,s} \geq 0$, $S^{b}_{B,s} > S^{ab}_{B,s} + N_{ac}$ *and* $S^{bc}_{B,s} \geq N_{cb} - N_{bc}$.

$S^{ab}_{B,s} \geq 0$ *and* $S^{ac}_{B,s} \geq 0$ implies a chosen.

$S^{b}_{B,s} > S^{ab}_{B,s} + N_{ac} \iff 2S^{b}_{B,s} > S^{a}_{B,s} + N_{ac}$

$\iff S^{b}_{B,s} + N_{ba} + N_{bc} > S^{a}_{B,s} + N_{ac}$

$\iff S^{b}_{B,s} + N_{ba} + N_{bc} + N_{ab} > S^{a}_{B,s} + N_{ac} + N_{ab}$

$\iff S^{b}_{B,s} + n + N_{bc} > S^{a}_{B,s} + N_{ab} + N_{ac}$

$\iff 4(n_5 + n_6) + 3n_1 + 2n_4 + n_2 + n_3 > 4(n_1 + n_2) + 2(n_3 + n_6)$ (1)

$S^{bc}_{B,s} \geq N_{cb} - N_{bc} \iff S^{b}_{B,s} + n + N_{bc} \geq S^{c}_{B,s} + n + N_{cb}$

$\iff 4(n_5 + n_6) + 3n_1 + 2n_4 + n_2 + n_3 > 4(n_3 + n_4) + 3n_2 + 2n_5 + n_1 + n_6$ (2)

The conditions (1) and (2) correspond to the fact of having yet 6 preference types but with 5 candidates as Borda gives now 4 points for a top candidate. We know that the number of candidates and the number of points given by each voter of type n_t for a, b and c so we find that the two new candidates d and e are always ranked respectively after b and c in the individual preferences. We conclude that d and e are respectively the WSC of b and c. Borda rule is sensitive to simultaneous manipulation by introduction of a WSC. The other cases are similar and we omitt them. Q.E.D.

The lemmas 1 and 2 give a complete characterization of the voting situations where the manipulators are able to win after manipulation by introduction of a WSC (single or simultaneous). Let $CM(n)$ be the set of these voting situations. The conditions that characterize $CM(n)$ can be translated in terms of the n_t's. We assume each voting situation to be equally likely to occur (this is the Impartial Anonymous Culture condition used by Gehrlein and Fishburn (1976)). Under this assumption, it is possible to obtain a polynomial representation for the number of elements in $CM(n)$ as a function of n and write the polynomial as follows: $|CM(n)| = x_5n^5 + x_4n^4 + x_3n^3 + x_2n^2 + x_1n + x_0$. Huang and Chua (2000) proved that there exists a periodicity e for the sequence of these polynomials [5], and provide an algorithm in order to find the periodicity and the coefficients of the different polynomials $CM(n)$. Thus, for 6 values of n $(= r+e,\ r+2e,\ r+3e,\ r+4e,\ r+5e)$, the same polynomial will give the number of voting situations where the manipulation can occur. We use computer enumeration to evaluate exact values for the number of the situations characterized by Lemmas 1 (resp. Lemmas 2) for each $n = 1$ to 145 (resp. 91). We set up 6 equations with 6 unknowns $(x_0, x_1, ..., x_5)$ and solve the 6 simultaneous equations. The algorithm allows us to find that the periodicity e is 24 (resp. 15) for single (resp. simultaneous) manipulation. Finally, we divide the cardinality of $CM(n)$ by the total number of situations, and obtain the following representations for the vulnerability of the Borda rule to this manipulation.
V(B^1, r(e)) (resp. V(B^2, r(e))) is the vulnerability of Borda rule to single (resp. simultaneous) manipulation with n $\equiv$ r(mod e). We give only the first polynomial.

Proposition 1

$$V(B^1, 1(24)) = \frac{1909n^5 + 24420n^4 + 111750n^3 + 208580n^2 + 94221n - 440880}{3072(n+1)(n+2)(n+3)(n+4)(n+5)}$$

$$V(B^2, 1(15)) = \frac{1023n^5 + 12900n^4 + 58745n^3 + 109260n^2 + 32280n - 214208}{1875(n+1)(n+2)(n+3)(n+4)(n+5)}$$

[5] See Huang and Chua (2000) for a general proof.

Table 2: Vulnerability of the Borda rule to manipulation by introduction of a WSC

n	B^1	n	B^2
1	0	1	0
2	0.4286	2	0.2857
3	0.3571	3	0.3571
4	0.4127	4	0.3333
5	0.4524	5	0.4087
6	0.4675	6	0.4026
7	0.4735	7	0.4090
8	0.5012	8	0.4312
9	0.5055	9	0.4380
10	0.5115	10	0.4469
11	0.5240	11	0.4560
12	0.5294	12	0.4584
13	0.5346	13	0.4628
14	0.5399	14	0.4711
15	0.5445	15	0.4753
16	0.5490	16	0.4776
17	0.5530	:	:
18	0.5552	:	:
19	0.5584	:	:
20	0.5622	:	:
21	0.5642	:	:
22	0.5656	:	:
23	0.5687	:	:
24	0.5707	:	:
25	0.5722	:	:
⋮	⋮	⋮	⋮
⋮	⋮	91	0.5318
145	0.6121	⋮	⋮
⋮	⋮	⋮	⋮
∞	0.6214	∞	0.5456

4 Concluding remarks

Our results (see Table 2) show the great vulnerability of the Borda rule to this manipulation. When compared with the manipulation attempted by voters misrepresenting their individual preferences, we see Borda being more vulnerable to manipulation by introduction of a WSC than voters manipulation.

In an election contest of three candidates when n is large the vulnerability of the Borda rule to manipulation by a coalition of strategic voters is about 50% (see Favardin, Lepelley, Serais 2002) whereas it is 62% (resp. 55%) for single (resp. simultaneous) manipulation by introduction of a WSC. These results reinforce the intuition of Dummett about the importance of this manipulation. However, this manipulation is a very specific case of manipulation by changing the choice set. If one doesn't impose specific conditions over the entering candidate, it could be the case that the Borda rule may be more robust than other positional rules. This conjecture is inspired by Gehrlein and Fisburn (1980) who looked at the likelihood of a positional rule to give the same ranking of candidates when one modified the choice set by eliminating one (resp. one or two) losing candidate from the choice set in the case of a three (resp. four) candidate election. They proved that, in each case, the Borda rule is the positional rule which maximizes the probability of giving the same ranking after the modification of the choice set. So, if we reverse this way of thinking, it can be conjectured that Borda maximizes the probability of giving the same ranking when adding a losing candidate to a choice set of two or three candidates.

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