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Mind the gap! A note on the income and the substitution effects

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Abstract

In this note, I study how the magnitude of the substitution and the income effects affects the change in the demand for a good following an own-price variation in a single consumer choice model with multiple goods.

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1. Introduction

Let us consider a simple two-period consumption-saving problem. A consumer is endowed with $\bar{x}_1 > 0$ units of income in the first period and has no endowment in the second period. As he or she can save at the interest rate r, his or her budget constraint can be written as $x_1 + s = \bar{x}_1$ and $x_2 = (1+r)s$, where s is saving and subscripts refer to the time period. Given a utility function $u(x_1, x_2)$ satisfying standard assumptions, it is possible to characterize the optimal level of consumption in the two periods as a function of the interest rate r, and to study how saving changes with it. As it is well known, this change depends on the interaction between the substitution and the income effect. To quote a prototypical description of the interplay between these two effects,

> intuitively, a rise in r has both an income effect and a substitution effect. The fact that the tradeoff between consumption in the two periods has become more favorable for second-period consumption tends to increase saving (the substitution effect), but the fact that a given amount of savings yields more second period consumption tends to decrease saving (the income effect)[1].

When the assumptions of the model are further specified by assuming a constant elasticity of substitution (CES) utility function, the above description is followed by the conclusion that if the elasticity of substitution is higher than one, then the substitution effect dominates the income effect and therefore saving increases as the interest rate raises, while the opposite happens if the elasticity of substitution is lower than one. Finally, if the elasticity of substitution is equal to one, saving is independent of r.

Although this reasoning is perfectly correct and intuitively appealing, it seems to leave open a gap between the standard tool it implicitly refers to, the Slutsky equation, and the conclusions it reaches. Upon reflection, it is not immediately obvious how it is possible to deduce the behavior of demand from the magnitude of a single parameter such as the (constant) elasticity of substitution. Moreover, the CES utility function belongs to the class of functions that represents homothetic preferences. It seems therefore natural to shed light the role this assumption plays in driving the above result. Finally, it is clear that the example just described belongs to a larger class of individual choice problems, namely those where a consumer can purchase $N \geqslant 2$ different goods by selling some of the only one is endowed with. Indeed, the traditional labor-leisure choice problem does also belong to this class[2]. In this regard, while restricting the analysis to the case $N = 2$ is suitable for an highly aggregated model, for more disaggregated models it may be more appropriate to assume $N > 2$. It is therefore useful to extend the analysis of the interplay between the substitution and the income effects to this case.

In this note, I will tackle these issues by using the Slutsky equation to express the own price elasticity of the only good the consumer is endowed with in terms of elasticities of substitution with respect to the other goods and income elasticity. In this way, it will be possible to explicitly link the properties of preferences to the different patterns of changes in

[1] See Romer (2001), p. 78.

[2] See e.g. Deaton and Muellbauer (1980), pp. 86-93.

demand. To the best of my knowledge, the characterization I propose here is original, and it can be seen as an extension to the case $N > 2$ of that by Atkinson and Stiglitz (1989, pp. 73-77)[3]. In the following section, I present the main result, an illustrative example and a brief concluding discussion.

2. The main result

Consider a consumer with preferences represented by a utility function $u(x)$ satisfying standard hypotheses, where x is a vector of consumption goods with generic entry x_n. The consumer is endowed with $\bar{x}_1 > 0$ of good 1. Let p denote the vector of prices for consumption goods with generic entry p_n, and consider the following problem:

$$\max_{x} \; u(x) \quad \text{s.t.} \quad p \cdot x \leqslant p_1 \bar{x}_1 \,. \tag{1}$$

Let $x_n(p, w)$ denote the demand for good n at prices p and income $w = p_1\bar{x}_1$. Moreover, let $h_n(p, u)$ be the compensated demand for good n at prices p and utility u[4]. The natural starting point of my analysis is the following version of the Slutsky equation:

$$\frac{\mathrm{d}x_1(p,w)}{\mathrm{d}p_1} = \frac{\partial h_1(p,u)}{\partial p_1} + \frac{\partial x_1(p,w)}{\partial w}\left(\bar{x}_1 - x_1(p,w)\right), \tag{2}$$

with $u = u(x(p, w))$. Let $\varepsilon(p, w)$ denote the income elasticity of good 1:

$$\varepsilon(p,w) = \frac{\partial x_1(p,w)}{\partial w}\frac{w}{x_1(p,w)}\,.$$

After some manipulation, it is possible to rewrite (2) as follows:

$$\frac{\mathrm{d}x_1(p,w)}{\mathrm{d}p_1}\frac{p_1}{x_1(p,w)} = \frac{\partial h_1(p,u)}{\partial p_1}\frac{p_1}{x_1(p,w)} + \varepsilon(p,w)\left(1 - s_1(p,w)\right)\,, \tag{3}$$

where $s_1(p, w) = p_1x_1(p, w)/w$. Because $p_1 > 0$ and $x_1(p, w) > 0$ by assumption, (2) and (3) have the same sign and in what follows I will study that of the latter. Consider the following (Morishima) elasticity of substitution between good n and good m[5]:

$$\eta_{nm}(p,u) = -\frac{\partial \log\left(h_n(p,u)/h_m(p,u)\right)}{\partial \log\left(p_n/p_m\right)} = \frac{\partial h_m(p,u)}{\partial p_n}\frac{p_n}{h_m(p,u)} - \frac{\partial h_n(p,u)}{\partial p_n}\frac{p_n}{h_n(p,u)}\,. \tag{4}$$

Because $h_m(p, u(x(p, w)) = x_m(p, w)$, by multiplying both sides of (4) by $s_m = p_mx_m/w$, we get[6]:

[3] For the case $N = 2$, see also Salanié (2003), pp. 46-47.

[4] I assume that $x \in \mathbb{R}^N_+$, $(p, w) \in \mathbb{R}^{N+1}_{++}$, $u > u(0)$ and that, for $n = 1, ..., N$, $x_n(p, w)$ and $h_n(p, u)$ are differentiable functions such that $x_n(p, w) > 0$ and $h_n(p, u) > 0$ for all (p, w, u).

[5] See Blackorby and Russell (1981, 1989) for a discussion of the properties of this type of elasticity of substitution.

[6] For simplicity, in what follows I omit the arguments of s_n.

$$\eta_{1m}(p,u)s_m + \left(\frac{\partial h_1(p,u)}{\partial p_1}\frac{p_1}{h_1(p,u)}\right)s_m = \frac{\partial h_m(p,u)}{\partial p_1}\frac{p_1 s_m}{h_m(p,u)} = \frac{\partial h_m(p,u)}{\partial p_1}\frac{p_m}{\bar{x}_1}\,,$$

where the last equality holds because $p_1 s_m / h_m(p,u)$ can be simplified to $p_m/\bar{x}_1$. Since $\sum_{m\neq 1} s_m = 1 - s_1$, by summing over $m \neq 1$, the above expression implies that:

$$\sum_{m\neq 1}\eta_{1m}(p,u)s_m + \left(\frac{\partial h_1(p,u)}{\partial p_1}\frac{p_1}{h_1(p,u)}\right)(1-s_1) = \sum_{m\neq 1}\frac{\partial h_m(p,u)}{\partial p_1}\frac{p_m}{\bar{x}_1}\,. \tag{5}$$

Because the second term on the left hand-side of the above expression can be simplified as follows:

$$\left(\frac{\partial h_1(p,u)}{\partial p_1}\frac{p_1}{h_1(p,u)}\right)(1-s_1) = \frac{\partial h_1(p,u)}{\partial p_1}\frac{p_1}{h_1(p,u)} - \frac{\partial h_1(p,u)}{\partial p_1}\frac{p_1}{\bar{x}_1}\,,$$

(5) can be written as:

$$\sum_{m\neq 1}\eta_{1m}(p,u)s_m + \frac{\partial h_1(p,u)}{\partial p_1}\frac{p_1}{h_1(p,u)} = \sum_{m=1}^{N}\frac{\partial h_m(p,u)}{\partial p_1}\frac{p_m}{\bar{x}_1} = 0\,, \tag{6}$$

where the last equality holds because $p \cdot S(p,u) = 0$, where S is the matrix of substitution effects. Finally, using (6), it is possible to rewrite (3) as follows:

$$\frac{\mathrm{d}x_1(p,w)}{\mathrm{d}p_1}\frac{p_1}{x_1(p,w)} = \sum_{m\neq 1}\left(\varepsilon(p,w) - \eta_{1m}(p,u)\right)s_m\,. \tag{7}$$

The above expression implies that:

$$\frac{\mathrm{d}x_1(p,w)}{\mathrm{d}p_1}\frac{p_1}{x_1(p,w)} \lesseqqgtr 0 \quad \text{if and only if} \quad \varepsilon(p,w) \lesseqqgtr \sum_{m\neq 1}\eta_{1m}(p,u)\hat{s}_m\,,$$

where $\hat{s}_m = s_m/(1-s_1)$. Therefore, what determines how demand of good 1 changes with its price is the magnitude of the income elasticity relative to a weighted sum of the elasticities of substitution between good 1 and the other goods. As an illustrative example, assume $N = 3$ and consider the following utility function, borrowed from Blackorby and Russell (1989):

$$u(x_1,x_2,x_3) = \min\left\{x_2, x_1^{0.5}x_3^{0.5}\right\}$$

In this case $\eta_{12} = 0.5$ and $\varepsilon = \eta_{13} = 1$ and therefore $\mathrm{d}x_1(p,w)/\mathrm{d}p_1$ is positive: demand increases with p_1 as the substitution effect with respect to good 3 is offset by the income effect, while that with respect good 2 is low as compared to the latter.

To better see how this result is related to the introductory example, assume $N = 2$ and let $(p_1,p_2) = ((1+r),1)$, so that $s_3 \equiv 0$ and the problem described at the beginning of the introduction is of the form (1). When preferences are homothetic, $x_n(p,w) = \tilde{x}_n(p)w$ and $h_n(p,u) = \tilde{h}_n(p)u$ for some functions $\tilde{x}_n$ and $\tilde{h}_n$ of prices alone. This implies that $\varepsilon(p,w) = 1$ and that η, the elasticity of substitution between good 1 and 2, does not depend on u nor does s_n depend on w. Therefore, in this case (7) implies that:

$$\frac{\mathrm{d}x_1(p,w)}{\mathrm{d}p_1}\frac{p_1}{x_1(p,w)} \lesseqgtr 0 \quad \text{if and only if} \quad 1 \lesseqgtr \eta(p)\,,$$

This condition clearly shows that, when preference are homothetic, first-period consumption decreases with the interest rate, and therefore saving increases, if $\eta(p)$ is greater than one. Assuming a CES utility function adds the further simplification that the elasticity of substitution is constant and therefore the changes in savings, and similarly the changes in labor supply for the labor-leisure choice model, can be determined by comparing its value to one.

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