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Multi-stage Double Auctions With Many Bidders

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Abstract

Traditionally in a Double Auction (also known as Bilateral Trade) a seller and a buyer interact to sell an object. Earlier literature had shown that in such a situation no mechanism will guarantee efficiency, incentive compatibility, individual rationality, and balanced budget condition. In this note we will argue that if we "sufficiently" increase the number of buyers then there is a two stage mechanism which satisfies all the four conditions stated above.

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1 Introduction

Consider a situation where there is a seller and a buyer bidding for an indivisible good to be sold. The valuations of both the seller and the buyer are both private information. The valuation of the seller is denoted by C and is distributed over the the interval $[\underline{c}, \overline{c}]$. Similarly the valuation of the buyer is denoted by V and distributed over the interval $[\underline{v}, \overline{v}]$. Both the valuations are independently distributed. The distributions are common knowledge and have full support on the respective intervals. Also assume that $\underline{v} < \overline{c}$ and $\overline{v} > \underline{c}$, i.e. there may be a case where it is efficient not to trade. The question we are interested in is : Is there a mechanism that ensures trade when $V > C$? The next impossibility proposition is due to Myerson and Satterthwaite (1983) who showed that it is impossible to design a mechanism under the above circumstances which guarantees trade whenever $V > C$ holds and simultaneously satisfies three other conditions viz. incentive compatibility, individual rationality and balanced budget.

Proposition. *In a double auction problem, there is no mechanism that is efficient, incentive compatible, individually rational, and at the same time balanced budget.*

Here we will assume weakly balanced budget[1]. Myerson and Satterthwaite (1983) showed that under the above situation all the efficient mechanisms that satisfy incentive compatible and individually rationality run in budget deficit. So, here we are interested in non-negative budget surplus only.

2 The Large Double Auction

In this section, first we will show that the above impossibility proposition holds even if we assume weakly balanced budget rather than balanced budget, for a situation where there is a single buyer and a single seller. Next we will design a mechanism that ensures efficiency, incentive compatibility, individual rationality and weakly balanced budget conditions under "sufficiently" large competition. This means to say that if the number of buyers are "sufficiently" large, then the mechanism satisfies all the four conditions stated above.

Proposition 2.1. *In a double auction problem, there is no mechanism that is efficient, incentive compatible, individually rational, and at the same time weakly balanced budget.*

Proof. Note that in this case the VCG mechanism runs in deficit. But since the VCG mechanism runs in deficit, every other mechanism also runs in deficit[2]. Thus, there does not exist an efficient mechanism that is incentive compatible, individually rational and simultaneously weakly balanced budget. □

Now we formulate a mechanism that will ensure efficiency, incentive compatibility, individual rationality and weakly balanced budget conditions under sufficiently large number of buyers. This

[1] A mechansim satsifies weakly balanced budget condition if and only if the expected gain of the mechanism designer is non-negative in that mechansim.

[2] See Krisna (2002), page 76, proposition 5.5

is a two stage mechanism where in the first stage each buyer bids for the object and pays his/her own bid and an entry fee. The mechanism designer selects the highest bidder. In the second stage, the highest bidder in the first stage and the seller bid. The buyer gets a particular amount from the mechanism designer depending on his/her bid. If in the second stage the bid of the buyer is more than that of the seller then only trade takes place, where the buyer pays what he/she bids to the seller and gets the object.

The idea behind such a mechanism is to compensate for the deficit of the mechanism designer by charging all the bidders in the first stage. Formally the mechanism is discussed below. For simplicity we assume that both the seller's and the buyer's valuations are distributed over the same interval, i.e. $\underline{v} = \underline{c}$ and $\overline{v} = \overline{c}$.

2.1 Assumptions

- There are N buyers, where $N > 1$ and a single seller.
- Seller has an object and wants to sell that object.
- The valuation of the seller is V_0. Valuation of the i^{th} buyer is V_i for all $i = 1, ..., N$.
- All the buyer's valuations and the seller's valuation lie in the interval $[0, \omega]$ i.e. $\forall j = 0, 1, ..., N$ $V_j \in [0, \omega]$. The valuations are distributed with a distribution function $F(.)$ which is increasing. $F(.)$ admits a continuous density $f(.) \equiv F'(.)$ and has full support.
- Both the seller and the buyers are risk neutral i.e. they want to maximize their respective expected profits.
- Buyers are not subject to any liquidity constraint.
- All components of the model other than the valuations are assumed to be commonly known to all the bidders, the seller and the mechanism designer.

2.2 The Mechanism

Stage I:

- The mechanism designer sets a reserve price of amount R, for the good, all the buyers have to submit a sealed bid more than or equal to that reserve price.
- The mechanism designer collects the bids from all the bidders (it is an all-pay auction with reserve price).
- The mechanism designer locates the highest bidder, and all the bidders except the highest bidder are out of the game.

Stage II:

- The highest bidder bids again for the object. Let he/she bid b_H^{II} in this stage.
- The mechanism designer pays the amount $\int_0^{b_H^{II}} F(t)dt$ to the highest bidder.
- The mechanism designer asks the seller that whether the seller is interested in selling the object at a price b_H^{II}. If the seller says "yes" then the highest bidder pays the seller an amount b_H^{II} and takes the object from the seller, otherwise no transaction takes place.

We will first derive the expected payoffs of the bidders, the objective of each bidder is to maximize her own payoffs. We will, then, derive the equilibrium bidding strategy for the seller, and finally we will derive the equilibrium bidding strategies of each bidder. Let the mechanism designer set a reserve price R. Let i^{th} bidder be the highest bidder in stage I. Also let her bid b_H^I in Stage I and b_H^{II} in Stage II. The payoff functions of the bidder are as follows:

Payoff function in Stage II:

$$\Pi_i^{II} = \begin{cases} (V_i - b_i^{II}) + \int_0^{b_i^{II}} F(t)dt & if\, the\, seller\, says\, YES \\ \int_0^{b_i^{II}} F(t)dt & Otherwise \end{cases}$$

Payoff function in Stage I:

$$\Pi_i^I = \begin{cases} (V_i - b_i^{II}) + \int_0^{b_i^{II}} F(t)dt - b_i^I & if\ b_i^I > \max_{j\neq i} b_j^I \ and\, the\, seller\, says\, YES\, in\, Stage\, II \\ \int_0^{b_i^{II}} F(t)dt - b_i^I & if\ b_i^I > \max_{j\neq i} b_j^I \ and\, the\, seller\, says\, NO\, in\, Stage\, II \\ -b_i^I & if\ b_i^I < \max_{j\neq i} b_j^I \end{cases}$$

The expected payoffs of the bidder is given below:

Expected payoff in Stage II: $(V_i - b_i^{II})F(b_i^{II}) + \int_0^{b_i^{II}} F(t)dt$

Expected payoff in Stage I: $\left[(V_i - b_i^{II})F(b_i^{II}) + \int_0^{b_i^{II}} F(t)dt\right] G(\beta^{-1}(b_i^I)) - b_i^I$

Note that $G(\beta^{-1}(b_i^I)) = F(\beta^{-1}(b_i^I))^{N-1}$, i.e. the probability that b_i^I is the highest bid at Stage I. Let us assume, V_R is a valuation, such that no bidder with a valuation less than V_R will participate in this auction. We will derive V_R formally after we calculate the equilibrium bidding strategies of a buyer in both the stage.

We will derive three propositions below, with proposition 2.2 telling us the equilibrium bidding strategy of the seller in stage II, proposition 2.3 focusing on the equilibrium bidding strategy for the highest bidder of Stage I in Stage II, and finally the proposition 2.4 deriving the equilibrium bidding strategy for each buyer in Stage I.

Proposition 2.2. *In the Stage II the equilibrium strategy of the seller is to accept the offer of the mechanism designer if $V_0 \leqslant b_H^{II}$ and reject the offer otherwise.*

Proof. Trivial. □

Proposition 2.3. *It is always optimal for the highest bidder of Stage I, to bid her true valuation in the Stage II.*

Proof. See appendix I. □

Proposition 2.4. *In stage I the equilibrium bidding strategy of the i^{th} bidder, whose valuation isV_i, is given by*

$$\beta(V_i) = \int_0^{V_i} F(t)dtG(V_i) - \int_{V_R}^{V_i} F(y)G(y)dy$$

Proof. See appendix II. □

Note that in Stage I

$$\begin{aligned}\Pi(V_i) &= \int_0^{V_i} F(t)dtG(V_i) - \beta(V_i) \\ &= \int_0^{V_i} F(t)dtG(V_i) - \left(\int_0^{V_i} F(t)dtG(V_i) - \int_{V_R}^{V_i} F(y)G(y)dy\right) \\ &= \int_{V_R}^{V_i} F(y)G(y)dy > 0\end{aligned} \tag{2.1}$$

as, at Stage II, we have already shown that, bidding her own valuation is the equilibrium bidding strategy for the highest bidder at Stage I.

Note that proposition 2.4 implies that the bidding strategy is incentive compatible[3], and the last equality of equation 2.1 guarantees that the bidding strategy is individually rational[4].

So, V_R is unique because $\Pi'(V_i) = F(V_i)G(V_i) > 0$ *as* $V_i \geqslant V_R > 0$. Now, once V_R is unique, we can derive an expression which guarantees all the four conditions stated above.

Proposition 2.5. *If N satisfies the expression below then the auction would satisfy all the four conditions.*

$$N \geqslant \frac{\int_{V_R}^{\omega} \int_0^t F(y)\,dy\,f(t)\,dt}{\left[R\left[1 - F(V_R)\right] + \int_{V_R}^{\omega} \int_{V_R}^{z} \int_0^y F(t)g(y)\,dt\,dy\,f(z)\,dz\right]} \tag{2.2}$$

Proof. See appendix III. □

Below we are going to provide an example.

[3] For a bidder, in Stage I, it is optimal to bid according to her own valuation since it maximizes her expected payoff. So the equilibrium bidding strategy is a function of her own valuation alone (note that as R is given to all the bidders V_R is also given to all the bidders) and the bidder has no incentive to submit a bid which is not a function of her true valuation.

[4] The expected gain from participating in the auction is strictly positive.

Example 2.6. Let us now assume that all the buyer's valuations and the seller's valuation are uniformly distributed over the interval $[0, \omega]$. First we will derive the equilibrium bidding strategy of a buyer in Stage I when her valuation is V_i.

$$\begin{aligned}
&\beta(V_i) = R + \int_{V_R}^{V_i} \int_0^y F(t)g(y)\,dt\,dy \\
&= R + \int_{V_R}^{V_i} \int_0^y \left\{\frac{t}{\omega}\right\} \frac{(N-1)}{\omega} \left[\frac{y}{\omega}\right]^{N-2} dt\,dy \\
&= R + \frac{(N-1)(V_i - V_R)^{N+1}}{2(N+1)\omega^N}
\end{aligned}$$

Let us now calculate the expected revenue of the mechanism designer from this auction.

$$\begin{aligned}
&R\left[1 - F(V_R)\right] + \int_{V_R}^{\omega} \int_{V_R}^{z} \int_0^y F(t)g(y)\,dt\,dy f(z)\,dz \\
&= R\left[1 - F(V_R)\right] + \int_{V_R}^{\omega} \left\{\frac{(N-1)(z-V_R)^{N+1}}{2(N+1)\omega^N}\frac{1}{\omega}\right\} dz \\
&= R\left[1 - \frac{V_R}{\omega}\right] + \frac{(N-1)(\omega - V_R)^{N+2}}{2(N+1)(N+2)\omega^{N+1}}
\end{aligned}$$

The expected loss of the mechanism designer is given by the equation below:

$$\begin{aligned}
&\int_{V_R}^{\omega} \int_0^t F(y)\,dy f(t)\,dt \\
&= \int_{V_R}^{\omega} \int_0^t \frac{y}{\omega}\,dy \frac{1}{\omega}\,dt \\
&= \frac{\omega^3 - V_R^3}{6\omega^2}
\end{aligned}$$

Therefore, the required number of buyers N must satisfy the following condition:

$$\begin{aligned}
N &\geqslant \frac{\frac{\omega^3 - V_R^3}{6\omega^2}}{R\left[1 - \frac{V_R}{\omega}\right] + \frac{(N-1)(\omega - V_R)^{N+2}}{2(N+1)(N+2)\omega^{N+1}}} \\
&= \frac{\omega^2 + \omega V_R + V_R^2}{6R\omega + \frac{3(N-1)(\omega - V_R)^{N+1}}{(N+1)(N+2)\omega^{N-1}}}
\end{aligned}$$

Note that if the above condition holds then the mechanism will satisfy all the four conditions stated above. For example let $\omega = 1$ and $R = 0.1$, so for any $N \geqslant 4$ the above condition is satisfied.

3 Conclusion

In this note we first show that even if we consider a weakly balanced budget condition as against a balanced budget condition, and if there is a single buyer and a single seller, no mechanism can guarantee efficiency, incentive compatibility, individual rationality and weakly balanced budget simultaneously. Finally we design a mechanism with sufficiently intense competition (i.e. if the number of buyers is sufficiently large) ensures efficiency along with incentive compatibility, individual rationality and weakly balanced budget.

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Appendix I

As we derived above, the expected payoff of the highest bidder of Stage I, in Stage II is given by:

$$(V_i - b_H^{II})F(b_H^{II}) + \int_0^{b_H^{II}} F(t)\,dt \tag{3.1}$$

The objective of the highest bidder is to maximize this expected payoff by choosing b_H^{II}. Differentiating the equation 3.1 with respect to b_H^{II} and setting that is equal to zero we get the first order condition.

$$\begin{array}{l}(V_i - b_H^{II})f(b_H^{II}) - F(b_H^{II}) + F(b_H^{II}) = 0 \\ OR, \qquad V_i = b_H^{II}\end{array}$$

The second order condition can be checked routinely.

Appendix II

Note that, the expected payoff of the highest bidder of the Stage I, is $\int_0^{V_i} F(t)\,dt$ at Stage II , where let the i^{th} bidder be the highest bidder in Stage I. As we have stated earlier that we define V_R in such a way that any bidder whose valuation is less than V_R will not enter into the auction and because the equilibrium profit function of a bidder is continuous, we must have

$$\left[\int_0^{V_R} F(t)\,dt\right] G(V_R) - R = 0 \tag{3.2}$$

In other words we must have $\beta(V_R) = R$ and $V_R > 0$. In stage I, therefore, at equilibrium the expected payoff of the i^{th} bidder is

$$\left[\int_0^{b_i^{II}} F(t)\,dt\right] G(\beta^{-1}(b_i^I)) - b_i^I$$

The objective of the i^{th} bidder is, therefore, to maximize this expected payoff by choosing b_i^I such that $b_i^I \geqslant R$. We already know that at the equilibrium if i^{th} bidder is the highest bidder then $b_i^{II} =$

V_i . Therefore at Stage I, the objective of the i^{th} bidder is to maximize $\left[\int_0^{V_i} F(t)\,dt\right] G(\beta^{-1}(b_i^I)) - b_i^I$ such that $b_i^I \geqslant R$ holds. Maximizing this with respect to b_i^I yields the first order condition:

$$\left[\int_0^{V_i} F(t)\,dt\right] \frac{g(\beta^{-1}(b_i^I))}{\beta'(\beta^{-1}(b_i^I))} - 1 = 0$$

where $g \equiv G'$.

At a symmetric equilibrium, $b_i^I = \beta(V_i)$ and as we assume (for the time being) that β is a strictly increasing function, we have $\beta^{-1}(b_i^I) = V_i$, and since $\beta(V_R) = R$, we have

$$\beta(V_i) = R + \int_{V_R}^{V_i} \int_0^y F(t) g(y)\,dt\,dy$$

Note that this bidding strategy is increasing in valuations.The second order condition for the minimization can be checked routinely. Finally note that

$$\begin{aligned}
&R + \textstyle\int_{V_R}^{V_i} \int_0^y F(t) g(y)\,dt\,dy \\
&= R + \textstyle\int_0^{V_i} F(t) dt G(V_i) - \left[\int_0^{V_R} F(t)\,dt\right] G(V_R) - \int_{V_R}^{V_i} F(y) G(y) dy \\
&= \textstyle\int_0^{V_i} F(t) dt G(V_i) - \int_{V_R}^{V_i} F(y) G(y) dy
\end{aligned}$$

The first equality is obtained by integrating by parts and as $\left[\int_0^{V_R} F(t)\,dt\right] G(V_R) = R$

Appendix III

The expected revenue of the mechanism designer is given by the expression below

$$\begin{aligned}
&N \textstyle\int_{V_R}^{\omega} \beta(z) f(z)\,dz \\
&= N \left[R\,[1 - F(V_R)] + \int_{V_R}^{\omega} \int_{V_R}^{z} \int_0^y F(t) g(y)\,dt\,dy f(z)\,dz\right]
\end{aligned}$$

The expected payment of the mechanism designer is given by the expression below $\int_{V_R}^{\omega} \int_0^t F(y)\,dy f(t)\,dt$

The weekly balanced budget condition for the mechanism designer will be satisfied if the condition below holds.

$$N \left[R\,[1 - F(V_R)] + \int_{V_R}^{\omega} \int_{V_R}^{z} \int_0^y F(t) g(y)\,dt\,dy f(z)\,dz\right] \geqslant \int_{V_R}^{\omega} \int_0^t F(y)\,dy f(t)\,dt$$