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Manilulation via endowments in university-admission problem

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Abstract

We consider a two-sided many-to-one matching model where universities offer scholarships to students. We show that every stable matching rule is manipulable by a university via destroying endowments under a fairly wide class of scholarship rules. Furthermore, we show that the set of Nash equilibria of the destruction game and the set of stable matchings may be disjoint.

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1 Introduction

Sertel and Özkal-Sanver (2002) study matching problems with endowments for one-to-one matching models under given monotonic consumption rules. They analyze the manipulability of optimal matching rules via endowments and show that men (respectively, women) can manipulate the women- (men-) optimal matching rule by destroying (hence hiding) as well as predonating their endowments. Furthermore, they show that the men- (women-) optimal matching rule is non-manipulable via hiding (hence destroying) by a man (woman) under all monotonic consumption rules. They characterize the classes of consumption rules under which optimal matching rules can be manipulated via destruction, hiding, and perfect hiding while Fiestras-Janeiro et al. (2004) characterize it for predonation. Atlamaz and Klaus (2006) study the manipulation via endowments in exchange markets with indivisible goods. Afacan (2011) studies application fee manipulations.

We carry this analysis to many-to-one matching problems, in particular to the university admission problem[1]. Interestingly, any stable matching rule is manipulable by universities via destroying and via their endowments under a fairly wide class of exogenous scholarship rules. Furthermore, we define a destruction game. We show that the set of Nash equilibria of the destruction game and the set of stable matchings may be disjoint.

2 Basic Notions

We take as given two nonempty, finite and disjoint sets $S = \{s_1, s_2, ..., s_k\}$ and $U = \{u_1, u_2, ..., u_l\}$, where $|S| = k \geq 3$ and $|U| = l \geq 2$. Let $A = S \cup U$ be the set of agents. Here, S stands for a set of students and U for a set of universities. By convention, we say that a student is assigned to the fictitious university $u_0 \notin U$ whenever he/she is assigned to no university. We assume that there are sufficient number of students, so that all universities fill their quotas.

For each agent $i \in A$ the set of potential mates of i, denoted by $A(i)$, is defined as

$$A(i) = \begin{cases} 2^S \backslash \{\emptyset\} & \text{if } i \in U \\ U \cup \{u_o\} & \text{if } i \in S. \end{cases}$$

A university $u \in U$ admits as many students as its capacity q_u which is a positive integer. By convention, we have $q_{u_o} = k$. Moreover, we have $q_u \geq 2$ for some $u \in U$. We denote $q = (q_{u_o}, q_{u_1}, ..., \ q_{u_l})$ by a **capacity vector.**

A **matching** $\mu : S \to U \cup \{u_0\}$ is a function such that, for all $s \in S, \mu(s) = \{u\}$ for some $u \in U \cup \{u_0\}$ and $\# \{s \in S \mid u = \mu(s) \leq q_u\}$ for all $u \in U \cup \{u_0\}$. We denote an inverse relation $\mu^{-1} : U \cup \{u_o\} \to S$ as $\mu^{-1}(u) = \{s \in S \mid u = \mu(s)\}$ for all $u \in U \cup \{u_o\}$. Let $\mathcal{M}(A)$ denote the set of all matchings for A.

Each university $u \in U \cup \{u_o\}$ has a non-negative **endowment** $e_i \in \Re_+$, whereas students have no endowment. By convention, we have $e_{u_0} = 0$. We regard $e \equiv (e_i)_{i \in U \cup \{u_o\}} \in \Re_+^{l+1}$ as an endowment profile.

Each student $s \in S$ consumes a pair $z_s = (u, m)$ which consists of a university $u \in U \cup \{u_o\}$ and some amount of money $m \in \Re_+$. Let $\Lambda = U \cup \{u_o\} \times \Re_+$ denote the set of all such university-money pairs. We assume that each student s has a complete and transitive

[1]An extended abstract of this paper took place in the *Proceedings of the Workshop on Rationality and Knowledge*, Artemov and Parikh, eds., 2006.

preference, denoted by R_s, over Λ, satisfying the following properties: For any $s \in S$, any two universities $u, u' \in U \cup \{u_o\}$, any amount of money $m, m' \in \Re_+$, **(i)** $(u', m)\ R_s\ (u, m)$ if and only if $(u', 0)\ R_s\ (u, 0)$ and **(ii)** $(u, m')\ R_s\ (u, m)$ if and only if $m' \geq m$. Let P_s and I_s respectively denote the strict and indifference relations associated with the preference relation R_s. Let R^S denote the preference profile of students.

Each university has a complete, transitive and antisymmetric preference relation over individual students.[2] Furthermore, we assume that each university has a responsive preference relation R_u on $2^S \backslash \{\emptyset\}$ to its preferences over individual students, in the sense that for any two assignments that differ in only one student, a university prefers the assignment containing the more preferred student (Roth (1985)).[3] Let $R^U = (R_u)_{u \in U}$ be the preference profile. Let P_u and I_u respectively denote the strict relation associated with the preference relation R_u.

Fixing the society A, the capacity vector q, the preference profile of students R^S and the preference profile of universities R^U, we refer $e \in \Re_+^{l+1}$ as a **(matching) problem** and the quadruplet $\left(A, q, R^S, R^U\right)$ as the **environment.** We assume that scholarships are distributed according to some exogenous scholarship rules $h : U \times \Re_+ \times S \to \Re_+$. In other words, under a scholarship rule h, each university u offers each student s some of its endowment e as scholarship which is denoted by $h_{su}(e)$. Let H be the class of exogenous scholarship rules satisfying the following properties:

1. h is announced before the matching occurs and is independent of the matching incurred.

   For all $u \in U$, for all $q_u \in \Re_+$, for all $e_u \in \Re_+$, and

2. for all $R_u \in \Re_+$, $h_{su}(e) \geq 0$ for all $s \in S$ and for all $e_u \in \Re_{++}$, $h_{su}(e) > 0$ for some $s \in S$,

3. for all $s, s' \in S$, $h_{su}(e) > h_{s'u}(e)$ implies $s\ P_u\ s'$,

4. for all $S' \subset S$ with $|S'| \leq q_u$, $\sum_{s \in S'} h_{su}(e) \leq e_u$,

5. for any $s \in S$ with $h_{su}(e) > 0$ and for any e' with $e'_u < e_u$ and $e'_{-u} = e_{-u}$, $h_{su}(e') < h_{su}(e)$.

6. There exists some $u \in U$ with capacity $q_u \geq 2$, some preference profile R_u and endowment $e_u \in \Re_{++}$ such that $h_{su}(e) > 0$ where s is ranked by u as the second best student.

The first property is crucial for the formation of students' preferences over university-money pairs before they apply to the universities. The second property states that each university offers nonnegative amount of scholarship to students and at least to one student a strictly positive amount. The third property states that each university allocates its

[2] In our model, universities do not gain any utility from money they hold to themselves. Most of the funds raised by universities are to be given specifically as scholarship and the amount of scholarships given to the students are about 0,1% of the universities total expenses.

[3] Take any university $u \in U$, any subset $S^* \subset S$ and any two students $\widehat{s}, s' \in S \diagdown S^*$ such that $\widehat{s}\ P_u\ s'$. For any responsive preference, we have $(S^* \cup \widehat{s})\ P_u\ (S^* \cup s')$.

endowment among the students consistent with its preference ordering. The fourth property is an ex-ante feasibility condition, which states that the amount of scholarship the university allocates to the matched students cannot exceed the amount of endowment it has. The fifth property is a resource monotonicity condition, which states that a student, who receives some positive amount of scholarship, receives less if the endowment of the university is lowered. The last property rules out the universities, which have quotas more than one, to give scholarship to only one student.[4]

Given the environment $\left(A, q, R^S, R^U\right)$, consider any endowment $e \in \Re_+^{l+1}$. A matching μ is individually rational if no student is assigned to a university that is worse than the no-university option. Formally, a matching $\mu \in \mathcal{M}$ is **individually rational** for $e \in \Re_+^{l+1}$ if for all $s \in S$, $\mu(s)\ R_s\ u_o$. A university $u \in U$, and a student $s \in S$ who is not assigned to u at some matching $\mu \in \mathcal{M}$ can block the matching μ under some scholarship rule h, if university u prefers s to some of its assigned students at μ and student s prefers being assigned to u and having a scholarship $h_{su}(e)$ to his/her present assignment and scholarship. Formally, a matching $\mu \in \mathcal{M}$ is **blocked by the university-student pair** $(u, s) \in U \times S$ at $e \in \Re_+^{l+1}$ under the scholarship rule h if $(u, h_{su}(e))\ P_s\ (\mu(s), h_{s\mu(s)}(e))$ and $s\ P_u\ s^*$ for some $s^* \in \mu^{-1}(u)$. A matching $\mu \in \mathcal{M}$ is **stable** for $e \in \Re_+^{l+1}$ under the scholarship rule h if it is individually rational for $e \in \Re_+^{l+1}$ and there is no university-student pair blocking at $e \in \Re_+^{l+1}$. Let $\mathcal{M}^*(e, h)$ be the set of all stable matchings for e under h. Given any scholarship rule h, a **matching rule** φ associates with each $e \in \Re_+^{l+1}$ a matching μ. Given any scholarship rule h, a **stable matching rule** φ associates with each $e \in \Re_+^{l+1}$ some stable matching $\mu \in \mathcal{M}^*(e, h)$.

3 Results

Let $\left(A, q, R^S, R^U\right)$ be an environment. A matching rule φ is said to be **manipulable via destroying endowments by a university** under some scholarship rule h if and only if there exist two endowments e and e' with $e'_{u^*} < e_{u^*}$ for some $u^* \in U$ and $e'_u = e_u$ for all $u \in U \backslash \{u^*\}$ such that $\varphi[e']^{-1}(u^*)\ P_{u^*}\ \varphi[e]^{-1}(u^*)$.

Proposition 3.1 *There exists an environment $\left(A, q, R^S, R^U\right)$ such that all stable matching rules are manipulable via destroying endowments by a university under any exogenous scholarship rule $h \in H$.*

Proof. Let $h \in H$. Let φ be any stable matching rule. Let $S = \{s_1, s_2, s_3\}$, $U = \{u_1, u_2\}$. Let $q = (q_{u_0}, q_{u_1}, q_{u_2}) = (3, 1, 2)$. Let $s_3\ Pu_2\ s_1\ Pu_2\ s_2$. Let $e_1 = 0$ and $e_2 > 0$ such that

[4]The equal rule, denoted by $h^=$, where each university u offers each student the same portion of its endowment as scholarship, and defined formally as $h^=_{su} = \frac{e_u}{q_u} \in \Re_+$ for all $s \in S$ and all $u \in U \cup \{u_0\}$, belongs to this class H. It is the unique exogenous scholarship rule which satisfies the ex-post efficiency condition, i.e., for all $u \in U$, for all $e_u \in \Re_+$, we have $\sum_{s \in C} h_{su}(e) = e_u$ for any $C \subset S$ with $|C| = q_u$.

$h_{s_1u_2}(e) > 0$. Consider the preference system represented below:

Ps_1	Ps_2	Ps_3	Pu_1	Pu_2
(u_2, e_2)	(u_2, e_2)	...	$\{s_1, s_2, s_3\}$	$\{s_1, s_2, s_3\}$
...	...	$(u_1, 0)$	$\{s_1, s_3\}$	$\{s_1, s_3\}$
$(u_2, h_{s_1u_2}(e))$	$(u_2, 0)$	...	$\{s_1, s_2\}$	$\{s_2, s_3\}$
...	...	(u_2, e_2)	$\{s_2, s_3\}$	$\{s_1, s_2\}$
$(u_1, 0)$	...	...	$\{s_1\}$	$\{s_3\}$
...	...	$(u_2, 0)$	$\{s_3\}$	$\{s_1\}$
$(u_2, 0)$	$(u_1, 0)$	...	$\{s_2\}$	$\{s_2\}$

The matching μ assigning s_1 and s_2 to u_2, and s_3 to u_1, is the unique stable matching for e under any $h \in H$. Hence, $\varphi[e] = \mu$. Let $e' = (0, 0, e'_2) = (0, 0, 0) \in \Re^2_+$. The matching ν assigning s_1 to u_1 , s_2 and s_3 to u_2, is the unique stable matching for e' under any $h \in H$. Hence, $\varphi[e'] = \nu$. Thus, u_2 is better off under any stable matching rule φ after destroying some of its endowment. ■

We define a destruction game as follows: Given any endowment $e \in \Re^{l+1}_+$, each university $u \in U \cup \{u_0\}$ has a strategy $d_u \in [0, e_u] = D_u$.[5] Write $D = \prod_{U \cup \{u_0\}} D_u$ for the set of strategy profiles. Every destruction $d \in D$ of endowments induces a new endowment $e(d) = e - d$. Given any $d, d' \in D$, we write $d \succeq_u d'$ if and only if $\varphi[e(d)]^{-1}(u)\ R_u\ \varphi[e(d')]^{-1}(u)$. Let $\succeq^U$ denote the set of preference profiles over D. Taking a stable matching rule φ, we construct a **destruction game** $(D, \varphi, \succeq^U)$ where φ is applied to $e(d)$. Given any endowment $e \in \Re^{l+1}_+$, a strategy profile $d \in D$ is a **Nash equilibrium of the game** $(D, \varphi, \succeq^U)$ if for all $u \in U$ and all $d' \in D$ with $d'_{-u} = d_{-u}$, we have $d \succeq_u d'$.[6] Let $N(D, \varphi, \succeq^U) \subseteq D$ denote the set of Nash equilibria of $(D, \varphi, \succeq^U)$. Let $\mathcal{N}(D, \varphi, \succeq^U) = \bigcup_{d \in N(D,\varphi,\succeq^U)} \{\varphi[e(d)]\} \subseteq \mathcal{M}$ denote the set of Nash equilibrium outcomes of $(D, \varphi, \succeq^U)$.

Proposition 3.2 *Let $h \in H$ be any scholarship rule. Let φ be any stable matching rule. There exist an environment (A, q, R^S, R^U) and an endowment $e \in \Re^{l+1}_+$, where the set of stable matchings and the (nonempty) set of Nash equilibria of the destruction game are disjoint, i.e. there exist (A, q, R^S, R^U) and $e \in \Re^{l+1}_+$ s.t. $\mathcal{N}(D, \varphi, \succeq^U) \cap \mathcal{M}^*(e, h) = \emptyset$ and $\mathcal{N}(D, \varphi, \succeq^U) \neq \emptyset$.*

Proof. Let $h \in H$. Let φ be any stable matching rule. Let $S = \{s_1, s_2, s_3\}$, $U = \{u_1, u_2\}$. Let $q = (q_{u_0}, q_{u_1}, q_{u_2}) = (3, 1, 2)$. Let $e = (e_0, e_1, e_2) \in \Re^3_+$ be such that $e_1 < h_{s_1u_2}(e) \leq e_2$.

[5] Since the fictitious university does not hold any endowments, we have $D_{u_0} = 0$.

[6] Let d_{-u} denote a strategy profile of all universities except the university u.

Consider the preference system represented below:

Ps_1	Ps_2	Ps_3	Pu_1	Pu_2
(u_2, e_2)	(u_2, e_2)	(u_1, e_1)	$\{s_1, s_2, s_3\}$	$\{s_1, s_2, s_3\}$
...	...	...	$\{s_1, s_3\}$	$\{s_1, s_3\}$
$(u_2, h_{s_1u_2}(e))$	$(u_2, 0)$	$(u_1, 0)$	$\{s_1, s_2\}$	$\{s_2, s_3\}$
...	...	...	$\{s_2, s_3\}$	$\{s_1, s_2\}$
(u_1, e_1)	(u_1, e_1)	(u_2, e_2)	$\{s_1\}$	$\{s_3\}$
...	...	...	$\{s_3\}$	$\{s_1\}$
(u_2, e_1)	$(u_1, 0)$	$(u_2, 0)$	$\{s_2\}$	$\{s_2\}$

Let μ be the matching assigning s_1 and s_2 to u_2, and s_3 to u_1, and ν the matching assigning s_1 to u_1, s_2 and s_3 to u_2. We have $\varphi[e] = \mu = \mathcal{M}^*(e,h)$. Note that for all $d \in D$, we have $(u_2, h_{s_2u_2}(e(d)))\, Ps_2\, (u_1, h_{s_2u_1}(e(d)))$ and $(u_1, h_{s_3u_1}(e(d)))\, Ps_3\, (u_2, h_{s_3u_2}(e(d)))$. Let also

$$D^1 = \{d \in D \mid (u_2, h_{s_1u_2}(e(d)))\, Ps_1\, (u_1, h_{s_1u_1}(e(d)))\},$$
$$D^2 = \{d \in D \mid (u_1, h_{s_1u_1}(e(d)))Ps_1\, (u_2, h_{s_1u_2}(e(d)))\},$$
$$D^3 = \{d \in D \mid (u_1, h_{s_1u_1}(e(d)))\, Is_1\, (u_2, h_{s_1u_2}(e(d)))\}.$$

Note that μ is the unique stable matching for $e(d)$ where $d \in D^1$, similarly v is the unique stable matching for $e(d)$ where $d \in D^2$. Furthermore, μ and v are the stable matchings for $e(d)$ where $d \in D^3$.

First, we show that $\nu \in \mathcal{N}(D, \varphi, \succeq^U)$. Take any $d \in D^2 \cup D^3$ such that $\varphi[e(d)] = \nu$. Suppose one of the universities, call it $\overline{u}$, changes its strategy and destroys some $\overline{d}_{\overline{u}} \in D_{\overline{u}}$. Either, we have $\varphi\left[e\left(\overline{d}\right)\right] = \nu$, both universities are as well off as before, or we have $\varphi\left[e\left(\overline{d}\right)\right] = \mu$ and both universities u_1 and u_2 get worse off. Hence, we have $\nu \in \mathcal{N}(D, \varphi, \succeq^U)$.

To show that $\mathcal{N}(D, \varphi, \succeq^U) \cap \mathcal{M}^*(e,h) = \emptyset$, it suffices to show that $\mu \notin \mathcal{N}(D, \varphi, \succeq^U)$. Take any $d \in D^1 \cup D^3$ such that $\varphi[e(d)] = \mu$. Since $(u_2, h_{s_1u_2}(e(d)))\ Rs_1\ (u_1, h_{s_1u_1}(e(d)))$, we have $h_{s_1u_2}(e(d)) > h_{s_1u_1}(e(d))$. Let u_2 change its strategy and destroy d_2' such that $(u_1, h_{s_1u_1}(e(d)))\ Ps_1\ (u_2, h_{s_1u_2}(e(d')))$ and $d' = (d_1, d_2') \in D^2$. We have $\varphi[e(d')] = \nu$. Since, u_2 is better off under ν then under μ, we have $\mu \notin \mathcal{N}(D, \varphi, \succeq^U)$. ■

4 Concluding Remarks

In the classical framework with no endowments, Roth (1985) proved that all stable matching rules are manipulable by a university via misrepresenting its preferences. In matching problems with endowments, however, universities -by changing the amount of scholarship they offer- affect students' preferences. More interestingly, under any stable matching rule, by offering a lower scholarship, a university may have more preferred students than before under a fairly wide class of exogenous scholarship rules.

Balinski and Sönmez (1999) model a student placement problem where preferences of universities are fictitious and based on the test scores of the students. They show that the university optimal mechanism is manipulable by students via underscoring in test scores. Test scores of the students can be interpreted as endowments of students.

We also show the existence of matching problems where the set of stable matchings and the set of Nash equilibria of the destruction game are disjoint. This result immediately implies the following two corollaries: The Nash equilibria of the destruction game may not be stable. Furthermore, we cannot produce every stable (hence individually rational) matching as an equilibrium of the destruction game. Now, we wish to compare these two results with the ones in classical university-admission problems with no endowments. Roth (1985) shows that the Nash equilibria of a preference revelation game (using a stable outcome function) may not be stable with respect to the true preferences. However, any individually rational matching with respect to the true preferences can be produced as an equilibrium of the preference revelation game using a stable outcome function.

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