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Brownian motion vs. pure-jump processes for individual stocks

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Abstract

Using recent activity signature function methodology developed in Todorov and Tauchen (2010), we provide empirical evidence that individual stocks from the New York Stock Exchange are adequately represented by a Brownian motion plus medium to large (rare) jumps thus invalidating the pure-jump process hypothesis proposed in numerous contributions. This result improves our understanding of the fine structure of asset prices and has implications for derivatives pricing.

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1 Introduction

We provide empirical evidence that highly-traded stocks may be modeled as Brownian motions plus rare medium to large jumps as advocated early in the financial literature (Merton, 1976). A more recent strand of the literature which argue that pure jump processes may also be used for asset price stochastic modelling is not supported by tick-by-tick data for our subsample of five stocks of the DJIA.

We use the recent methodology developed in Todorov and Tauchen (2010) which proposes to plot the Activity Signature Function (ASF) using intraday returns of discretely observed stochastic processes. The ASF methodology is nonparametric in essence and thus avoids the drawbacks of estimating parameters when volatility or jump risk premia have to be specified. This avoids assumptions about the form of these premia as noted in Bandi and Perron (2006, p. 647): “To explicitly account for a risk premium, the existing work relies on tight parametrizations for it.”

Our approach allows to investigate the presence of jumps of infinite activity. This is not the case for the nonparametric approach using recent bipower variations (BPV) measures proposed in Barndorff-Nielsen and Shephard (2004, 2006) (BNS hereafter) that are only robust to the presence of large but rare jumps. As emphasized in Andersen *et al.* (2007a, p. 128) using such measures: “The main limitation is that we exclude Lévy jump processes with infinite jump intensity, see, e.g., Carr *et al.* (2002), as we only allow for “rare” jumps occurring at a finite rate. Hence, key issues are to what extent the jump-diffusion representation is consistent with empirical data and what specific features of the specification are necessary in order to adequately describe the observed return processes.”

This limitation strongly advocates for the use of a methodology able to consider both finite and infinite activity jumps and, possibly, to distinguish between them. This is precisely the objective of the ASF. Hence, we shall be able to distinguish between what should be considered as pure diffusion, jump-diffusion or pure-jump processes.

To date, the ASF methodology has been applied to foreign exchange data and Internet

traffic data (Todorov and Tauchen, 2010) and to S&P 500 futures and the VIX from CBOE (Todorov and Tauchen, 2011). We are not aware of any use of the ASF for other series and in particular for individual stocks. Using the ASF methodology, this note investigates which stochastic processes are more plausible with data in hands.

Identifying jumps in a stochastic process is important because it has implications for risk management, option pricing, portfolio selection and also has consequences for optimal hedging strategies.¹ Indeed, when computed using simulation techniques, the quantiles sensibly differ when draws are from a continuous or continuous plus jump distribution. Similarly, portfolio selection can be dramatically modified when some assets in the investment universe are potentially jumping. Liu, Longstaff and Pan (2003) study the implications of jumps in both prices and volatility on investment strategies when a risk-free asset and a stochastic-volatility jump-diffusion stock are the available investment opportunities. Cvitanić, Polimenis and Zapatero (2008) propose a model where the asset returns have higher moments due to jumps and study the sensitivity of the investment in the risky asset to the higher moments, as well as the resulting utility loss from ignoring the presence of higher moments (see also Aït-Sahalia *et al.* (2009)).

As for option derivatives pricing, early attempts to include jump in stochastic models for security prices are Merton (1976), Ball and Torous (1983), Jorion (1988) and Kaushik (1993) among others. The authors show how to value derivative securities considering that a jump may arise following a given distribution for the jump size and a given distribution for jump arrivals rate. Note that stochastic volatility and jumps are not redundant features (see the discussion in Cont and Tankov (2004)). Indeed, while both these characteristics allow to better consider the excess kurtosis (fat-tails) present in the distribution of returns, they do not have the same impact on option prices (see Ball and Torous (1985), Bakshi *et al.* (1997), Cont and Tankov (2004), Jiang (2007) or Wu (2008)).

Our preliminary results point to the relevance of the Brownian motion for modelling the individual stock prices, at least for the five stocks under investigation. In addition, rare

¹The impact of jumps in returns and volatility is studied in Andersen *et al.* (2002), Eraker *et al.* (2003), Chernov *et al.* (2003), Eraker (2004), Broadie *et al.* (2007). A risk premium (jump risk premium) can be raised in reference to jumps (Pan, 2000).

jumps of medium to large size would help to fit the data better as note previously in the thorough analysis by Andersen *et al.* (2010). The authors rely on the nonparametric approach of BNS (2004, 2006) and Andersen *et al.* (2007b) and provide empirical evidence of the existence of jumps in individual stocks of the DJIA index. We complement their findings by using a method, the ASF, which is also suited for infinite activity jumps.

The plan for this paper is as follows. The next section presents briefly the methodology in Todorov and Tauchen (2010) and Section 3 provides a description of the data as well as our empirical results. The last section gives a number of implications of our results and some ideas for future research.

2 The activity signature function (Todorov and Tauchen, 2010)

The ASF methodology allows to associate an activity index to any continuous-time process. This is an extension of the Blumenthal-Gettoor (1951) index which is indicative of the vibrancy of a stochastic process but, contrary to the latter, the ASF is valid for all stochastic processes and not only for pure-jump processes.² In short, Todorov and Tauchen (2010) propose to compute the logarithm of the sum of power variations for different values of the power argument and show that the ratio of the this sum for two different sampling frequencies has interesting properties. In particular, the value of this ratio above two and below two allows to detect the presence of a Brownian component. Powers below two will give a stronger role to the continuous component while powers above two will emphasize the jump component. We refer the reader to the original article by Todorov and Tauchen (2010) for further details of the methodology which is based on limit theorems for power variations.³

We enhance the original ASF in Todorov and Tauchen (2010) by relying on a two-scale methodology as in Zhang *et al.* (2005). In short, we keep a sampling interval for intraday returns of 5 minutes as in in the bulk of the literature using this kind of data but we

²See also Aït-Sahalia and Jacod (2009) or Lee and Hannig (2010) who propose related tests.

³Todorov and Tauchen (2011) provide a very clear and more accessible presentation of their ASF when examining the stochastic process underlying the VIX index.

sample with starting points at each minute. Thus we have 5 more estimates of the ratio at each period and an average of these ratios provides much more robust results. To be more precise, let

$$V(p, \Delta_n) = \sum_{i=1}^n |r_i|^p$$

be the sum of intraday returns r_i in absolute value at the sampling frequency $1/n$ (conversely the sampling interval is Δ_n). The returns in absolute value are taken at a power p thereby the notation $V(p, \Delta_n)$. A particular case of power variations is when $p = 2$ which corresponds to the so-called realized variance. Todorov and Tauchen (2010) show that considering other values for p has interesting properties and suggest to compute the ratio

$$b(p, \Delta_n) = \frac{\ln(2)p}{\ln(2) + \ln[V(p, 2\Delta_n)] - \ln[V(p, \Delta_n)]}$$

for different values of p going from zero to four. For a given p , $b(p, \Delta_n)$ mainly compare the values of $V(p, \cdot)$ for two different sampling intervals Δ_n and $2\Delta_n$. This ratio is computed for different intervals of observations going from 1 day to 22 days (one trading month). Information about the stochastic process we are interested in can be inferred from quantiles of the distribution of these values. Todorov and Tauchen (2010) define the QASF as:

$$\mathcal{B}_\alpha(p, \Delta_n) = \text{Quantile}_\alpha[b(p, \Delta_n)]$$

Using the QASF allows to limit the impact of extreme realizations while keeping a reliable idea of the behavior of $b(p, \Delta_n)$ over the different periods of observations. More importantly, QASF plots will help to draw conclusions about the nature of the stochastic process which is likely to drive stock returns. Several cases are noteworthy. For a Brownian motion the activity index is two no matter the power p considered. In case

of a Brownian motion plus jump process, the activity index is two for $p < 2$ and p for $p > 2$. Finally, for a pure-jump process, the QASF should be decreasing for some powers $p \in (0, 2]$ thereby indicating a lower activity level than for a continuous semimartingale. The examination of the QASF thus allows to discriminate between several models of the stock returns dynamics.

3 Empirical findings

3.1 Data

Our data set includes five individual stocks from the New York Stock Exchange (NYSE): Citigroup Inc., General Electric Co., IBM, McDonald's Corp. and Walt Disney Co.. These are highly liquid stocks which are part of the DJIA Index. Data is extracted from the Trades and Quotes (TAQ) database of NYSE and are in the form of quotes (bid and ask) with time stamps. We only retain quotes recorded in the official trading period (9:30 EST to 16:00 EST) and compute mid-quotes as representative of the efficient price. We remove days with a shortened trading period thus resulting in a set of 1118 days over the period July 2, 2001 to December 30, 2005 for each stock.

3.2 Empirical findings

Figures of QASF for all five individual stocks are plotted in Figures 1 to 10. The horizontal axis is for the power p . As noted above, the interesting p to be considered are between zero and four. We thus plot the QASF on this interval. The vertical axis is for the QASF, i.e. the quantiles of the $b(p, \Delta_n)$ which provides a measure of the activity of the stochastic process as can be inferred from observed (discrete) intraday data. To enhance robustness of our results, for each stock, we provide eight representations of the QASF which is itself a plot of the lower and upper quartiles as well as the median. We let the sampling frequency to vary from 5 to 10 minutes and we adopt blocks of different lengths (1, 5, 10 and 22 days).

From Figures 1 to 10, we observe that the median is most of the time a flat function of the

power p at a constant value of two. As indicated in Section 2, this is expected for a very vibrant process such as a Brownian motion (see Todorov and Tauchen (2010) Theorem 1.(a)). The case of General Electric Co. is perhaps the less conclusive as the median is not as flat as it should be for a continuous semimartingale without, nevertheless, making this assumption unlikely.

Interestingly, the observation of the upper quartile indicates that in most cases, medium to large jumps are part of the process and should be considered when modeling individual shares. Indeed, for power p above two, the upper quartile is often increasing in p which is an indication of the impact of jumps. The intuition behind the argument for this conclusion is that larger increments (intraday returns) such as jumps have a larger impact when considered at a power $p > 2$. Importantly, this conclusion is valid for all the five stocks under investigation and confirms the findings in Andersen *et al.* (2010).

4 Implications and concluding remarks

Our empirical results point to the fact that a Brownian motion plus jumps could be a good proxy for modelling asset prices thereby invalidating underlying pure jump processes theories. This validate the early model of Merton (1976) which is an extension of the Black-Scholes framework allowing for (rare) jumps.

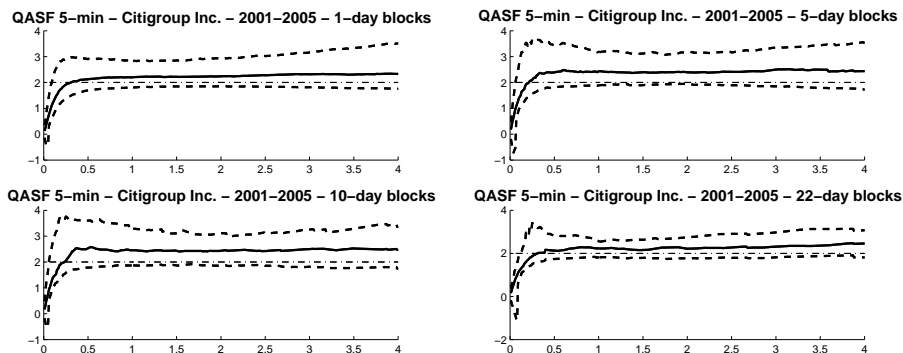
A first implication of this result is that pure-jump processes are not plausible candidates for modeling stock returns. Among many others, are the models of Barndorff-Nielsen (1997), Barndorff-Nielsen and Shephard (2001), the CGMY model by Carr *et al.* (2002). This is good news for derivatives pricing as in the presence of jumps, usual dynamic hedging arguments fail and markets are fundamentally incomplete (see Cont and Tankov, 2004) leading to complicated pricing formulae. The Merton's (1976) framework also leads to incompleteness but in a more tractable way than in a pure-jump process.

A second implication of our results relate to our understanding of the fine structure of asset prices. Indeed, microfounded theories in Geman *et al.* (2000) or Pakkanen (2010) among others, describe the price of assets as being pure jump processes. This result arises from

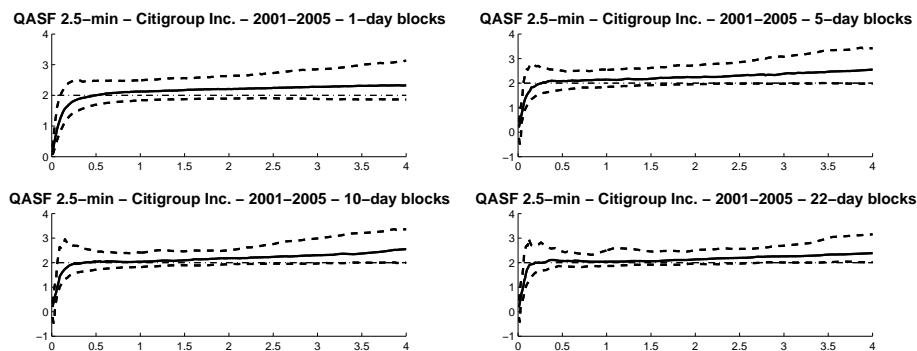
market clearing conditions, the arrival of news which is discrete by nature and the trading behavior of investors who trade on news. Our results thus point to the fact that trading activity in highly traded stocks such as the five individual stocks under investigation in the present paper is sufficient to deliver evidence of activity level of a Brownian motion. It has to be noted that our empirical results involve highly traded individual stocks and may not be generalized straightforwardly to less liquid shares. Indeed, following the theoretical analysis in Pakkanen (2010), one may wonder whether the trading activity in less liquid shares would be sufficient for the stochastic process to appear as a Brownian motion. This remains an empirical question which would necessitates a rigorous comparison of the ASF for shares of different liquidity and this work is beyond the scope of the present note.

Recent analysis by Cont and Mancini (2009) and Aït-Sahalia and Jacod (2010) complement the Todorov and Tauchen's (2010) framework. The main aim of these papers remain to use the jump activity index to discuss the relevancy of considering jumps to model asset returns and these methodologies may be used to confirm and extend our results. Nevertheless, we believe that the next research step would be to relate the level of trading activity to the activity index thereby enhancing further our understanding of the price formation of asset returns.

Figures

**Figure 1**

QASF for Citigroup Inc. for the period July 2, 2001 to December 29, 2005. QASF is computed using 5 and 10 minutes sampling intervals and blocks of 1, 5, 10 and 22 days.

**Figure 2**

QASF for Citigroup Inc. for the period July 2, 2001 to December 29, 2005. QASF is computed using 2.5 and 5 minutes sampling intervals and blocks of 1, 5, 10 and 22 days.

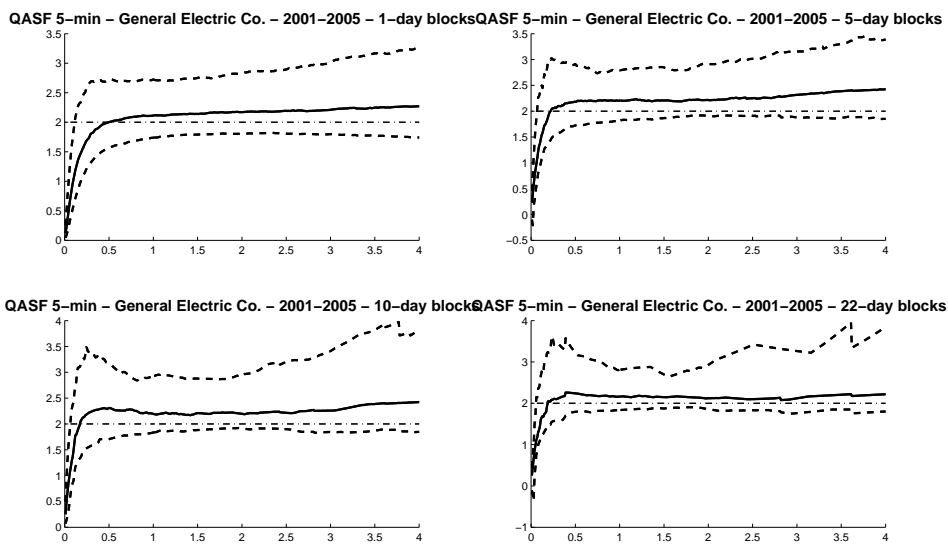


Figure 3
 QASF for General Electric Co. for the period July 2, 2001 to December 29, 2005. QASF is computed using 5 and 10 minutes sampling intervals and blocks of 1, 5, 10 and 22 days.

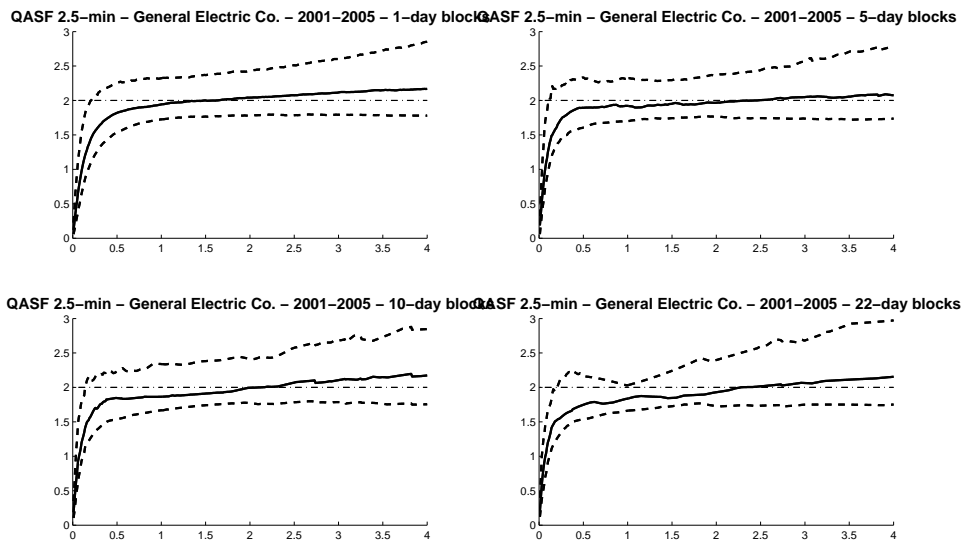


Figure 4
 QASF for General Electric Co. for the period July 2, 2001 to December 29, 2005. QASF is computed using 2.5 and 5 minutes sampling intervals and blocks of 1, 5, 10 and 22 days.

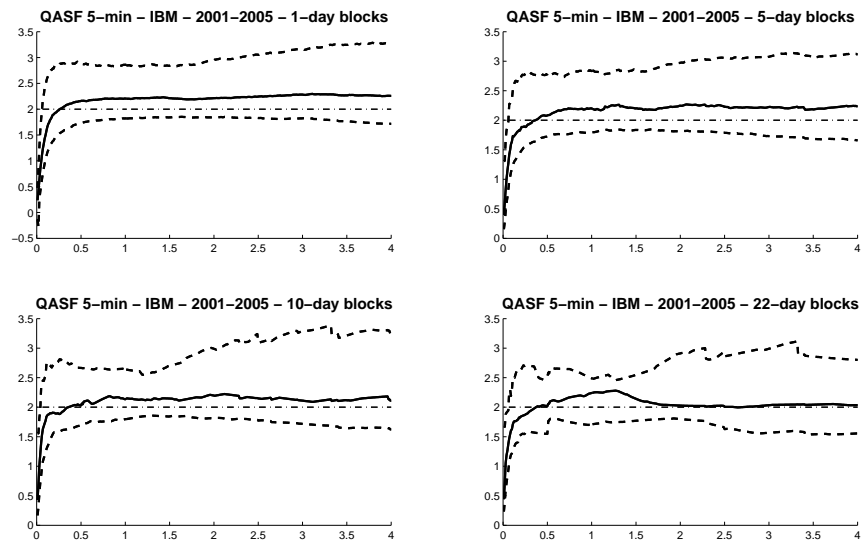


Figure 5
 QASF for IBM for the period July 2, 2001 to December 29, 2005. QASF is computed using 5 and 10 minutes sampling intervals and blocks of 1, 5, 10 and 22 days.

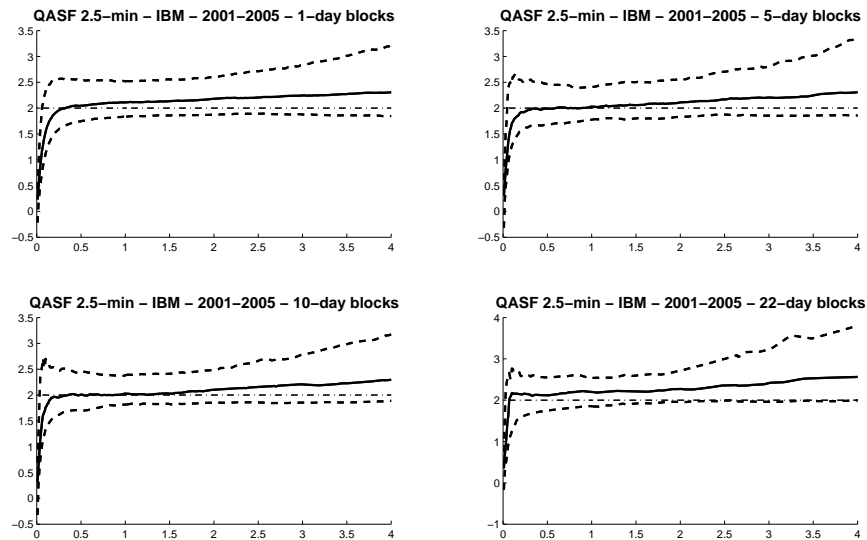


Figure 6
 QASF for IBM for the period July 2, 2001 to December 29, 2005. QASF is computed using 2.5 and 5 minutes sampling intervals and blocks of 1, 5, 10 and 22 days.

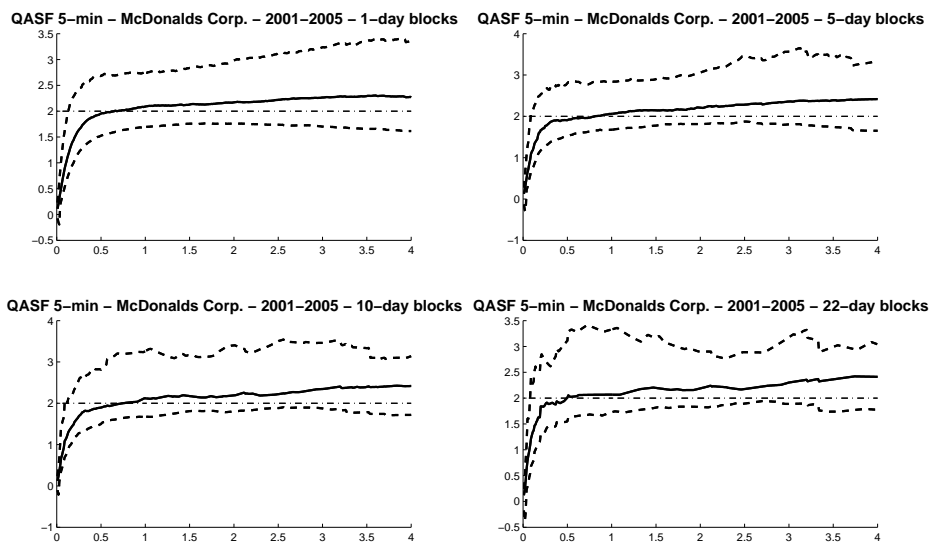


Figure 7
 QASF for McDonald's Corp. for the period July 2, 2001 to December 29, 2005. QASF is computed using 5 and 10 minutes sampling intervals and blocks of 1, 5, 10 and 22 days.

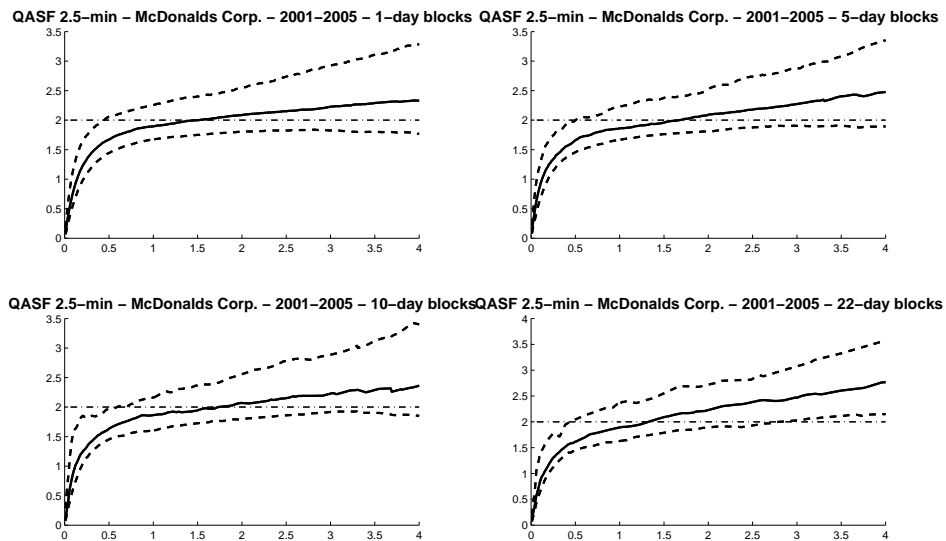


Figure 8
 QASF for McDonald's Corp. for the period July 2, 2001 to December 29, 2005. QASF is computed using 2.5 and 5 minutes sampling intervals and blocks of 1, 5, 10 and 22 days.

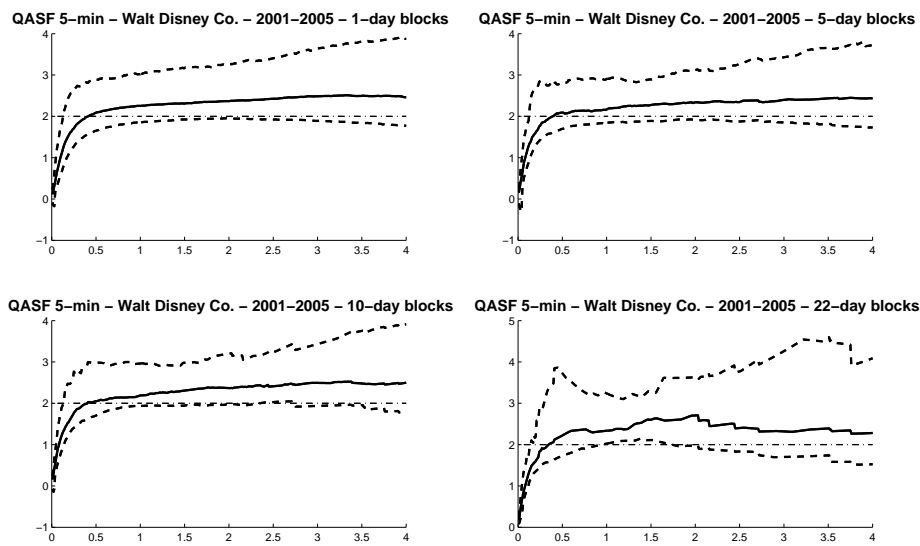


Figure 9
 QASF for Walt Disney Co. for the period July 2, 2001 to December 29, 2005. QASF is computed using 5 and 10 minutes sampling intervals and blocks of 1, 5, 10 and 22 days.

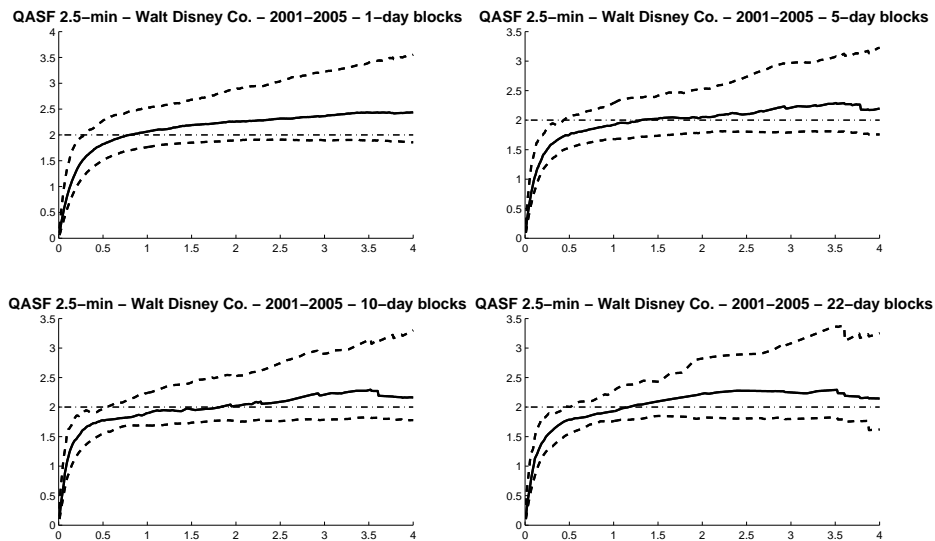


Figure 10
 QASF for Walt Disney Co. for the period July 2, 2001 to December 29, 2005. QASF is computed using 2.5 and 5 minutes sampling intervals and blocks of 1, 5, 10 and 22 days.

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