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Existence of speculative bubbles when time-horizons are finite

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Abstract

This note extends the existing literature on speculative bubbles by allowing for arbitrary trading sequences. As our main result we prove that bubbles may exist in a myopic rational expectations equilibrium (Radner 1979) if and only if every agent expects infinitely many trading opportunities to exist. For finite horizons our finding implies the possible existence of bubbles under the plausible bounded rationality condition that every agent believes he will not end-up with holding the asset when the bubble bursts.

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1. Introduction

The possible existence of asset pricing bubbles has been studied for dividend-payment- and trading processes characterized by time nodes $\{0, 1, ..., T\}$ where the time-horizon T is either finite or infinite.[1] Within an arbitrage pricing framework Santos and Woodford (1997) show that—under fairly general conditions—speculative bubbles can only exist for infinite time horizons (Theorem 3.1 in Santos and Woodford 1997). Within a *rational expectations equilibrium* (=REE) framework in the sense of Radner (1979)—relevant to the present paper—Tirole (1982) proves the seminal result that a myopic REE may support bubbles if and only if the time-horizon T is infinite. Moreover, Tirole (1982) also shows that any bubble must satisfy the discounted martingale property; that is, the expected value of a bubble has to increase forever in accordance with the representative agent's time-discount factor. These technical conditions—the infinite time-horizon combined with the discounted martingale property—are largely at odds with our perception of real-life bubbles as pyramid schemes that are bound to burst with certainty sooner rather than later. For example, even the most optimistic person at the height of the latest real-estate frenzy would have agreed that real-estate prices must move back to "normal" within, say, the next ten years.

In this note we investigate conditions under which an REE may support bubbles that will burst with certainty at a finite time-horizon T. To this end we consider general trading processes characterized by arbitrary increasing sequences of time nodes $\{t_n \mid n = 0, 1, ...\}$ such that $\lim_{n\to\infty} t_n = T$. Trading happens in our model between finitely many risk-neutral agents who share a common subjective prior about the asset's dividend-payment process. While we allow for heterogenous agents in the form of asymmetric information, our focus on *fully revealing* REEs implies that all agents share the same information at equilibrium prices. As a consequence, our approach gives rise to a uniquely defined bubble term as the difference between the equilibrium price and the asset's expected fundamental value. The expectation is thereby taken with respect to the unique (subjective) conditional probability measure incorporating all agents' information revealed at equilibrium prices.

As our main formal finding we establish that myopic fully revealing REEs may support bubbles if and only if $\max\{t_n \mid n = 0, 1, ...\}$ does not exist. This technical condition has a straightforward economic interpretation: Even if it is common knowledge to all agents that a bubble will burst with certainty at time T, there may exist a bubble at every trading node before T as long as every period t_k, $k = 0, 1, ...$, agent believes that

[1] Early contributions to this literature include Sargent and Wallace (1973), Blanchard (1979), and Flood and Garber (1979). For a more recent overview on the vast REE literature on bubbles see, e.g., Brunnermeier (2001).

all period $t_{k+1}, t_{k+2}, \ldots$ agents believe that they will have a trading opportunity before the time-horizon T finally arrives. Intuitively, a positive (negative) bubble may therefore exist in our REE framework because at every trading node the buyer (seller) assumes that he will be able to re-sell (re-buy) the asset in time. A typical trading sequence $\{t_n \mid n = 0, 1, \ldots\}$ supporting such beliefs for a finite time-horizon would be one where trading becomes faster and faster generating thereby infinitely many trading opportunities before "closing date" T.

Key to the existence of bubbles in our model is thus the agents' conviction in never-ending trading opportunities rather than an infinite time horizon per se. Beyond our mere technical result, our approach therefore gives rise to a behavioral interpretation of bubbles in terms of myopic agents who are "overconfident" in the following sense: Every agent believes in beating the other agents in the sense that he is certain to not be caught holding the asset when a positive bubble bursts, (to hold the asset when a negative bubble bursts, respectively).

2. Model

Consider the time set $[0, T]$ such that $T \in \mathbb{R}^+ \cup \{\infty\}$ and fix an arbitrary sequence of time nodes $\mathcal{T} = \{t_k \in [0, T] \mid k = 0, 1, \ldots\}$. We interpret the members of $\mathcal{T}$ as the time nodes at which trading between agents may happen. Let Ω denote the states of the world. For every time node $t \in \mathcal{T}$ we assume that the period t agents, denoted $i_t \in \{1, \ldots, n\}$, are risk-neutral and share the same prior π. Let Π_{i_t} denote each period t agent's private information partition of Ω and $\mathcal{F}_{i_t}$ the σ-algebra generated by Π_{i_t}, i.e. $\mathcal{F}_{i_t} := \sigma(\Pi_{i_t})$. Further, denote by $\mathcal{F}_t$ the σ-algebra generated by the *joint* (cf. Aumann 1976) of all period t agents' information partitions, i.e., $\mathcal{F}_t := \sigma(\Pi_t)$ such that

$$\Pi_t := \bigvee_{i_t \in \{1, \ldots, n\}} \Pi_{i_t}. \tag{1}$$

That is, Π_t describes the information partition that would result when all period t agents shared their private information with each other. We thus consider the probability space $(\Omega, \mathcal{F}_T, \pi)$ where π is an additive subjective probability measure and the σ-algebra $\mathcal{F}_T$ is generated by the sequence of sub σ-algebras $\{\mathcal{F}_{t_n}\}_{n=0,1,..}$. We further assume that, for all t_k, $\Pi_{t_{k+1}}$ is finer than Π_{t_k} so that $\{\mathcal{F}_{t_k}\}_{k=0,1,..}$ constitutes a filtration.

Let $\mathcal{D}$ be a space of $\mathbb{R}^+$-valued stochastic processes on $[0, T]$. We associate with a given asset the process $D \in \mathcal{D}$, i.e., $D : \Omega \times [0, T] \to [0, \infty)$ whereby we interpret $D(\omega, t)$ as the asset's dividend-payment at time t and state ω. We impose that this process is product measurable. That is, D is measurable with respect to the smallest sigma-algebra on $\Omega \times \mathcal{T}$ containing all sets of the form $A \times B$ where $A \in \mathcal{F}_T$ and B is in the Borel sigma-algebra on $[0,T]$. We also impose that D is an adapted processes, i.e., the random variable D_t is measurable with respect to $\mathcal{F}_t$.

Write $\Delta_{k+n}^{k} := t_{k+n} - t_k$ and denote by $e^{-\delta\Delta_{k+n}^{k}} \in (0,1)$ the agents' common time discount factor associated with time interval Δ_{k+n}^{k} . The time-discounted total dividend payments of the asset in time interval Δ_{k+1}^{k} are then given by the following Lebesgue integral

$$\int_{\mu>t_k}^{t_{k+1}} e^{-\delta(\mu-t_k)} D(\mu) d\mu. \tag{2}$$

Note that this definition of accumulated dividend-payments in terms of a Lebesgue integral allows us to incorporate general dividend-payment processes that do not necessarily have to be continuous almost everywhere as in the case of the Riemann integral.

According to myopic optimizing behavior, period t_k equilibrium prices must equal the expected value of the asset at period t_{k+1} whereby this expectation is conditional on the information received in period t_k. In an REE every agent's information at period t_k must also include all information revealed by the period t_k equilibrium price. In the case of a fully revealing REE this information revealed at equilibrium prices is common to all agents and equivalently described by the information partition (1). These—somewhat informal—considerations allow us to define the period $t \in \mathcal{T}$ fundamental value of the asset for all agents as the following $\mathcal{F}_t$-measurable random variable

$$f(I(t)) := \begin{cases} E\left(\int_{\mu>t}^{T} e^{-\delta(\mu-t)} D(\mu)\, d\mu \mid I(t)\right) & \text{if } t < T \\ 0 & \text{if } t = T \end{cases} \tag{3}$$

whereby the expectation is taken with respect to the subjective conditional probability measure $\pi(\cdot \mid I(t))$ such that $I(t) \in \Pi_t$.[2]

Definitions. Myopic fully-revealing REE; Speculative bubbles

- In a *myopic fully revealing REE equilibrium* prices are given as a $\mathbb{R}^{+}$-valued stochastic process $p : \Omega \times \mathcal{T} \to [0,\infty)$ where every period $t_k \in \mathcal{T}$ equilibrium price function is an $\mathcal{F}_{t_k}$-measurable random variable $p(t_k) : \Omega \to [0,\infty)$ such

[2] Observe that, by an application of Fubini's Theorem,

$$E\left(\int_{\mu>t}^{T} e^{-\delta(\mu-t_k)} D(\mu) d\mu \mid I(t)\right) = \int_{\mu>t}^{T} E\left(e^{-\delta(\mu-t_k)} D(\mu) \mid I(t)\right) d\mu$$

so that the integrand in (3) is measurable with respect to $\pi(\cdot \mid I(t))$. Consequently, (3) is well-defined.

that

$$p(t_k) \quad = \quad \begin{cases} E\left(e^{-\delta\Delta_{k+1}^k} p(t_{k+1}) + \int\limits_{\mu>t_k}^{t_{k+1}} e^{-\delta(\mu-t_k)} D(\mu) d\mu \mid I(t_k)\right) & \text{if } t_k < T \\ 0 & \text{if } t_k = T \end{cases} \tag{4}$$

with $I(t_k) \in \Pi_{t_k}$.

- The period t bubble-term is defined as the $\mathcal{F}_t$-measurable random variable

$$B\left(t\right) := p(t) - f\left(I\left(t\right)\right) \tag{5}$$

with $p(t)$ given by (4). Whenever $B\left(t\right)\left(\omega\right) \neq 0$ we say that there exists a *speculative bubble* at time t in state $\omega \in \Omega$.

By restricting attention to fully revealing REEs we are not losing much generality. Namely, recall that Radner (1979) shows for a large class of economies that fully revealing REEs generically exist and, moreover, that any REE is generically fully revealing. While Radner's proof requires—in contrast to our assumption of risk-neutrality—strictly risk-averse agents, our framework of linear utilities also covers Radner economies when we simply reinterpret π as *risk-neutral* subject probability measure (cf. Theorem 2 in Dybvig and Ross 2003).

3. Results

Theorem "Discounted martingale property". Fix an arbitrary increasing sequence of time nodes $\mathcal{T} = \{t_k \in [0, T] \mid k = 0, 1, ...\}$. In any myopic fully revealing REE it must hold that, for all $\forall k, n \in \mathbb{N}$ such that $t_{k+n} \in \mathcal{T}$,

$$B\left(t_k\right) = e^{-\delta\Delta_{k+n}^k} E\left(B\left(t_{k+n}\right) \mid I\left(t_k\right)\right). \tag{6}$$

Proof: Let M denote the μ-measurable sets. The key fact to note is that since the function $e^{-\delta(w-t)}D(w) : T \to [0, \infty]$ is measurable,

$$\phi(A) := \int\limits_A D d\mu \tag{7}$$

is a measure on M. For notational simplicity, we let D denote the function $e^{-\delta(w-t)}D(w)$. Thus, for any collection of countable disjoint measurable sets, $A_i \in M$, $i = 1, 2, ...,$ we have

$$\int_A D d\mu := \phi\left(\bigcup_{n\in\mathbb{N}} A_n\right) \tag{8}$$

$$= \sum_{n\in\mathbb{N}} \phi\left(A_n\right) \tag{9}$$

$$= \sum_{n\in\mathbb{N}} \int_{A_n} D d\mu \tag{10}$$

where

$$A := \bigcup_{n\in\mathbb{N}} A_n. \tag{11}$$

As we shall see, this property is key to the proof since, in general, Lebesgue integrals cannot be added together as easily as Riemann integrals.

Fix some $t_k \in \mathcal{T}$ and define the (half-open) time interval $A_{i,j}$ as follows

$$A_{i,j} := (t_{k+i}, t_{k+j}] \text{ for all } \forall t_{k+i}, t_{k+j} \in \mathcal{T}. \tag{12}$$

Note that $A_{i,j} \in M$, i.e., each $A_{i,j}$ is μ-measurable, since, by construction, all Borels are measurable. By the equilibrium condition (4),

$$p(t_k) = E\left(e^{-\delta\Delta_{k+1}^k} p\left(t_{k+1}\right) + \int_{A_{0,1}} D d\mu \mid I\left(t_k\right)\right) \tag{13}$$

$$= E\left(e^{-\delta\Delta_{k+1}^k} p(t_{k+1}) + \phi\left(A_{0,1}\right) \mid I\left(t_k\right)\right). \tag{14}$$

Suppose now that the inductive hypothesis is true:

$$p(t_k) = E\left(e^{-\delta\Delta_{k+m}^k} p(t_{k+m}) + \int_{A_{0,m}} D d\mu \mid I\left(t_k\right)\right) \tag{15}$$

$$= E\left(e^{-\delta\Delta_{k+m}^k} p(t_{k+m}) + \phi\left(A_{0,m}\right) \mid I\left(t_k\right)\right). \tag{16}$$

Applying the law of iterated expectations and using our assumption of an information filtration gives

$$p(t_k) = E\left(e^{-\delta\Delta_{k+m}^k} \cdot e^{-\delta\Delta_{k+m+1}^{k+m}} p(t_{k+m+1}) + \phi\left(A_{m,m+1}\right) + \phi\left(A_{0,m}\right) \mid I\left(t_k\right)\right) \tag{17}$$

$$= E\left(e^{-\delta\Delta_{k+m+1}^k} p\left(t_{k+m+1}\right) + \phi\left(A_{0,m+1}\right) \mid I\left(t_k\right)\right) \tag{18}$$

whereby the last step follows because ϕ is an additive measure and $A_{m,m+1} \cap A_{0,m} = \emptyset$. Hence, we deduce by induction on the time nodes in $\mathcal{T}$:

$$p(t_k) = E\left(e^{-\delta\Delta_{k+n}^k} p(t_{k+n}) + \phi\left(A_{0,n}\right) \mid I\left(t_k\right)\right) \text{ for all } \forall n \in \mathbb{N}. \tag{19}$$

Introducing the bubble-term (5) give for all $\forall k, n \in \mathbb{N}$ such that $t_{k+n} \in \mathcal{T}$

$$p(t_k) = E\left(e^{-\delta\Delta^k_{k+n}} p(t_{k+n}) + \phi\left(A_{0,n}\right) \mid I\left(t_k\right)\right) \tag{20}$$
$$= E\left(E\left(\phi\left(A_{n,\infty}\right) \mid I(t_{k+n})\right) \mid I(t_k)\right) \tag{21}$$
$$+ e^{-\delta\Delta^k_{k+n}} E\left(B(t_{k+n}) \mid I(t_k)\right) + E\left(\phi\left(A_{0,n}\right) \mid I(t_k)\right) \tag{22}$$
$$= E\left(\phi\left(A_{0,\infty}\right) \mid I(t_k)\right) + e^{-\delta\Delta^k_{k+n}} E\left(B(t_{k+n}) \mid I(t_k)\right) \tag{23}$$
$$= E\left(\int\limits_{\mu>t_k}^{T} D d\mu \mid I\left(t_k\right)\right) + e^{-\delta\Delta^k_{k+n}} E\left(B(t_{k+n}) \mid I(t_k)\right) \tag{24}$$

whereby (23) follows since ϕ is a measure and the family of measurable sets $A_{i,j}$ are disjoint. Thus,

$$B(t_k) = e^{-\delta\Delta^k_{k+n}} E\left(B(t_{k+n}) \mid I(t_k)\right) \text{ for all } \forall k, n \in \mathbb{N} \text{ such that } t_{k+n} \in \mathcal{T}, \tag{25}$$

i.e., bubbles are discounted martingales on the time nodes in $\mathcal{T}$. This proves the theorem.□□

Proposition. Fix an arbitrary sequence of time nodes $\mathcal{T} = \{t_k \in [0, T] \mid k = 0, 1, ...\}$ such that $\lim_{k\to\infty} t_k = T$. There may exist a speculative bubble at some time node $t \in \mathcal{T}$ if and only if $\max \mathcal{T}$ does not exist.

Proof. Part (i). Suppose that $\max \mathcal{T}$ exists. In that case $\max \mathcal{T} = T$. By (4), the period T equilibrium price is given as

$$p\left(T\right) = D(T) \tag{26}$$

so that the period T bubble-term becomes

$$\begin{aligned} B\left(T\right) &= p\left(T\right) - f\left(T\right) && (27)\\ &= D\left(T\right) - D\left(T\right) && (28)\\ &\Leftrightarrow \\ B\left(T\right)\left(\omega\right) &= 0 \text{ for } \forall \omega \in \Omega. && (29)\end{aligned}$$

By the above theorem, for all $t \in \mathcal{T}$ such that $t < T$,

$$\begin{aligned} B\left(t\right) &= e^{-\delta(T-t)} E\left(B\left(T\right) \mid I\left(t\right)\right) && (30)\\ &= e^{-\delta(T-t)} \cdot 0 && (31)\\ &\Leftrightarrow \\ B\left(t\right)\left(\omega\right) &= 0 \text{ for } \forall \omega \in \Omega. && (32)\end{aligned}$$

so that there cannot exist a speculative bubble at any time node $t \in \mathcal{T}$ and in any state $\omega \in \Omega$ when $\max \mathcal{T}$ exists.□

Part (ii). Suppose now that $\max \mathcal{T}$ does not exist. By the above theorem,

$$\begin{aligned} B(t_k) &= e^{-\delta\Delta_{k+1}^{k}} E\left(B(t_{k+1}) \mid I(t_k)\right) && (33)\\ &= e^{-\delta\Delta_{k+1}^{k}} E\left(e^{-\delta\Delta_{k+2}^{k+1}} \cdot E\left(B(t_{k+2}) \mid I(t_{k+1})\right) \mid I(t_k)\right) && (34)\\ &= e^{-\delta\Delta_{k+1}^{k}} \cdot e^{-\delta\Delta_{k+2}^{k+1}} E\left(B(t_{k+2}) \mid I(t_k)\right) && (35)\end{aligned}$$

whereby the last step follows from the law of iterated expectations and our assumption of an information filtration. Repeating this argument shows that, for any $t_k \in \mathcal{T}$ and any $n \in \mathbb{N}$,

$$\begin{aligned} B(t_k) &= e^{-\delta\left(\Delta_{k+1}^{k}+\Delta_{k+2}^{k+1}+\ldots+\Delta_{k+n}^{k+n-1}\right)} E\left(B(t_{k+n}) \mid I(t_k)\right) && (36)\\ &= e^{-\delta\Delta_{k+n}^{k}} E\left(B(t_{k+n}) \mid I(t_k)\right). && (37)\end{aligned}$$

Fix now a state $\omega \in \Omega$ such that there exists a speculative bubble at t_k, i.e.,

$$B(t_k)(\omega) = b \neq 0. \tag{38}$$

Then, for $\omega \in I(t_k)$,

$$E\left(B(t_{k+n}) \mid I(t_k)\right) = \frac{1}{e^{-\delta\Delta_{k+n}^{k}}} \cdot b \neq 0. \tag{39}$$

However, since

$$E\left(B(t_{k+n}) \mid I(t_k)\right) = \int_{\omega' \in \Omega} B(t_{k+n})\, d\pi\left(\omega' \mid I(t_k)\right) \tag{40}$$

for $\omega \in I(t_k)$, there must exist some set $A \subseteq I(t_k)$ in $\mathcal{F}_{t_{k+n}}$ with $\pi(A \mid I(t_k)) > 0$ such that

$$B(t_{k+n})(\omega') \neq 0 \text{ for all } \omega' \in A. \tag{41}$$

This proves that whenever there exists a speculative bubble at some $t_k \in \mathcal{T}$, then there also exists a speculative bubble with positive probability at all time nodes $t \in \mathcal{T}$ such $t_k < t$. Thus, by simply introducing a state $\omega \in \Omega$ in which a bubble exists at t_0, we construct the possible existence of a bubble at all time-nodes $t \in \mathcal{T}$ whenever $\max \mathcal{T}$ does not exist.□□

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