Economics Bulletin

Volume 32, Issue 1

A Note on the implementation of the Pareto efficient allocation in the Lagos-Wright model

Tao Peng
Southwestern University of Finance and Economics

Abstract

This note modifies Lagos – Wright (2005) by adding subsidies to sellers. We show that this modification can result in a Pareto efficient allocation at the Friedman rule when buyers do not have all the bargaining power. We find that the optimal rate of subsidy is increasing in buyers' relative risk aversion coefficient.

I thank Luis Araujo, Ricardo Lagos, Randall Wright and seminar participants at Missouri Economics Conference (2010) for useful comments. All remaining errors are mine.
Citation: Tao Peng, (2012) "A Note on the implementation of the Pareto efficient allocation in the Lagos-Wright model", *Economics Bulletin*, Vol. 32 No. 1 pp. 27-36.
Contact: Tao Peng - pengtao@swufe.edu.cn.
Submitted: August 29, 2010. **Published:** January 09, 2012.

1. Introduction

Lagos and Wright (2005) develop a monetary model in which the frictions that make money essential are explicitly modeled and at the same time, monetary policy can be analyzed in a tractable manner. In Lagos and Wright (2005), the Friedman rule may or may not result in a Pareto efficient allocation. In particular, when buyers have all the bargaining power, the economy can achieve the first best allocation. When buyers do not have all the bargaining power, the economy does not achieve the first best allocation.

There are two types of inefficiencies in the Lagos-Wright model. The first is due to the discount rate. In a monetary economy, holding money incurs cost. For an individual, the cost of holding one dollar balance is the cost of reducing a dollar of consumption, which depends on the individual's time preference or the discount rate. The second type of inefficiency is due to partial bargaining power. An agent who carries a dollar is making an investment, since he gives up current consumption for future consumption. If the buyer does not have full bargaining power, the seller will bargain away part of the surplus, which will result in less production. This is the so-called holdup problem. The Friedman rule corrects the first type of inefficiency by generating a real return on money that compensates for discounting, but it does not correct the second type of inefficiency.

Kocherlakota (2005) argues that a problem in the basic literature, the Lagos-Wright model included, is that all tax instruments beyond the inflation tax are eliminated from the environment. Since including additional tax/subsidy instruments may matter when understanding the nature of optimal monetary policy, Kocherlakota (2005) suggests that "it would be interesting to know to what extent the government could use other instruments like production subsidies or consumption taxes to cure the bargaining and search inefficiencies present in these setting." [1]

In this paper, we pursue Kocherlakota's suggestion. Precisely, we modify the Lagos –Wright model by adding a production-cost subsidy to sellers. We show that this modification can result in a Pareto efficient allocation under the Friedman rule even if buyers do not have all the bargaining power. Intuitively, subsidies provide incentives for sellers to produce more, offsetting the holdup effect induced by buyers' partial bargaining power. Using specific functional forms, we find that the optimal rate of subsidy is an increasing function of buyers' relative risk aversion coefficient.

Gomis-Porqueras and Peralta-Alva (2010) also study how production subsidies can be used to restore efficiency of equilibria in the Lagos-Wright model. They find that when lump-sum taxes are available and production subsidies in the decentralized market are available, the equilibria are efficient under the Friedman rule regardless of the value of the bargaining power of the buyer. A weakness of Gomis-Porqueras and Peralta-Alva (2010) is that due to the absence of record keeping in the decentralized market, the implementation of production subsidy requires that agents disclose their money holdings before they enter the decentralized market and after they leave the decentralized market. Agents that increase their money holdings (producers) are given a monetary subsidy. This creates the possibility that agents form a coalition to reach an agreement to pool money with the purpose of obtaining a subsidy, while production does not really increase. In this note, we follow Diamond (1982) to assume that the government has sufficient policy tools to control production decisions so that it can subsidize production directly. Thus, in terms of restoring efficiency of equilibria under the Friedman rule, this note can be seen as a complement of Gomis-Porqueras and Peralta-Alva (2010).

2. The model

The environment follows that of Lagos and Wright (2005). Time is discrete. There is a [0, 1] continuum of agents who live forever with discount factor $\beta \in (0,1)$. Each period consists of two subperiods: day and night. During the day, agents supply labor, trade with other agents in a decentralized market and consume. If trade happens, we assume that sellers will be subsidized by the government. At night, agents supply labor, trade with other agents in a centralized market, and consume. The agent's period utility function is given by

$$\widetilde{U}(x,h,X,H) = u(x) - [c(h) - s(h)] + U(X) - H \quad (1)$$

Where x and h (X and H) are consumption and labor during the day (night). Let $s(h)$ be the subsidy function, $s'(h) > 0$. As in Lagos and Wright (2005), in order to get the distribution of money degenerate, $\widetilde{U}$ is assumed to be linear in H. Assume that $u, c,$ and U are twice continuously differentiable with $u' > 0, c' > 0, U' > 0, u'' < 0, c'' \geq 0$ and $U'' \leq 0$. Also, $u(0) = c(0) = 0$ and suppose that there exists $q^* \in (0,\infty)$ such that $u'(q^*) = c'(q^*) - s'(q^*)$ and $X^* \in (0,\infty)$ such that $U'(X^*) = 1$ with $U(X^*) > X^*$.

During the day, in the decentralized market, an agent meets with another agent with probability α. The day good x comes in many varieties, but each agent consumes only a subset of them. Each agent can produce one unit of consumption good x with one unit of labor. For two agents i and j drawn at random, the probability that agent $i\,(j)$ consumes what $j\,(i)$ produces but not vice versa is σ. The probability that both consume what the other can produce is δ. And the probability that neither wants what the other produces is $1 - 2\sigma - \delta$.

At night, agents trade the general good X in the centralized market. Agents at night can transform one unit of labor into one unit of the general good.

There is another object, called money, which is intrinsically useless, perfectly divisible and serves as the medium of exchange in both the decentralized market and the centralized market. The net growth rate of the money supply is constant over time and equal to τ. New money is injected by lump-sum transfers or withdrawn by taxes if $\tau < 0$. These transfers or taxes take place during the night. The subsidy to sellers is funded by a lump-sum tax, call it T, which is equally split between buyers and sellers in the centralized market, or by money withdrawal if $\tau < 0$, or by both.

3. Equilibrium

We first consider the case where the total quantity of money is fixed at M. Let $F_t(\tilde{m})$ be the measure of agents starting the decentralized day market at t holding $m \le \tilde{m}$. Let $V_t(m)$ and $W_t(m)$ be the value functions for an agent with m dollars when he enters the decentralized and the centralized markets, respectively. In the decentralized market, we denote the amount transferred in a transaction by $q_t(m,\tilde{m})$ and use $d(m,\tilde{m})$ to denote the dollars the buyer pays, where m and $\tilde{m}$ are the buyer's and seller's money holdings. Let $B_t(m,\tilde{m})$ be the payoff for an agent holding m in double-coincidence meetings. Then, Bellman's equation is

$$\begin{aligned} V_t(m) = \alpha\sigma \int \{u[q_t(m,\tilde{m})] + W_t[m - d_t(m,\tilde{m})]\} dF_t(\tilde{m}) \\ + \alpha\sigma \int \{-[c(q_t(\tilde{m},m)) - s(q_t(\tilde{m},m))] + W_t[m + d_t(\tilde{m},m)]\} dF_t(\tilde{m}) \\ + \alpha\delta \int B_t(m,\tilde{m}) dF_t(\tilde{m}) + (1 - 2\alpha\sigma - \alpha\delta) W_t(m), \end{aligned} \tag{2}$$

With probability $\alpha\sigma$, two agents meet and carry out a trade with money. With probability $\alpha\delta$, two agents meet and they exchange their goods. With probability $1 - 2\alpha\sigma - \alpha\delta$, no trade occurs. The four terms in (2) represent the expected payoffs to buying, selling, bartering, and not trading.

The problem of the agent in the centralized market is to choose consumption, labor supply, and the money holdings carried to the next period to maximize his lifetime utility subject to the budget constraint. That is, the agent solves

$$\max_{X,H,m'} \{U(X) - H + \beta V_{t+1}(m')\} \tag{3}$$

subject to $X = H + \phi_t m - \phi_t m' - \frac{1}{2}T$, $X \ge 0, 0 \le H \le \overline{H}$ and $m' \ge 0$, where $\overline{H}$ is an upper bound on hours, m' is money taken out the market, and ϕ_t is the price of money in the centralized market. Substituting for H from the budget constraint, solving for X and rearranging terms, we obtain

$$W_t(m) = \phi m + W(0), \tag{4}$$

where $W(0) = U(X^*) - X^* + \max_{m'}\{-\phi m' + \beta V(m')\}$ and $U'(X^*) = 1$.

The terms of trade in the decentralized market is determined by bargaining. Assume that the buyer has bargaining power $\theta > 0$ and that threat points are given by continuation values $W_t(m)$. In double-coincidence meetings, as in LW (2004), we can show that symmetric Nash bargaining solution suggests that each agent produces q^* and no money change hands. Therefore, $B_t(m,\tilde{m}) = u(q^*) - [c(q^*) - s(q^*)] + W_t(m)$. In single-coincidence meetings, the generalized Nash solution suggests that the terms of trade are determined by

$$\max_{q,d} [u(q) + W_t(m-d) - W_t(m)]^{\theta} [-(c(q) - s(q)) + W_t(\tilde{m} + d) - W_t(\tilde{m})]^{1-\theta} \tag{5}$$

Subject to $d \le m$ and $q \ge 0$,

Substituting (4) into (5), the bargaining problem simplifies to

$$\max_{q,d}[u(q)-\phi\, d]^{\theta}[-(c(q)-s(q))+\phi\, d]^{1-\theta} \tag{6}$$

subject to $d \le m$ and $q \ge 0$

The constraint $d \le m$ states that the buyer cannot spend more than his money holdings. In equilibrium, as shown by Lagos and Wright (2005), $d = m$, and the quantity produced, denoted as $\hat{q}_t(m)$, solves the first order condition

$$\phi_t m = \frac{\theta[c(q)-s(q)]u'(q)+(1-\theta)u(q)[c'(q)-s'(q)]}{\theta\, u'(q)+(1-\theta)[c'(q)-s'(q)]} \equiv Z(q), \tag{7}$$

Since the solution does not depend on the seller's money holdings $\tilde{m}$, we can write $q_t(m,\tilde{m}) = q_t(m)$ and $d_t(m,\tilde{m}) = d_t(m)$.

Given the bargaining solution and expression (4), we can simplify (2) to

$$V_t(m) = v_t(m) + \phi_t m + \max_{m'}\{-\phi_t m' + \beta V_{t+1}(m')\}, \tag{8}$$

where

$$\begin{aligned} v_t(m) \equiv\ & \alpha\sigma\{u[q_t(m)] - \phi_t d_t(m)\} + \alpha\sigma\int\{\phi_t d_t(\tilde{m}) - [c(q_t(\tilde{m})) - s(q_t(\tilde{m}))]\}dF_t(\tilde{m}) \\ & + \alpha\delta\{u(q^*) - [c(q^*) - s(q^*)]\} + U(X^*) - X^* \end{aligned} \tag{9}$$

By repeated substitution we have

$$V_t(m_t) = v_t(m_t) + \phi_t m_t + \sum_{j=1}^{\infty}\beta^{j-1}\max_{m_{j+1}}\{-\phi_j m_{j+1} + \beta[v_{j+1}(m_{j+1}) + \phi_{j+1} m_{j+1}]\}. \tag{10}$$

Applying the analysis of the properties of the objective function in Lagos and Wright (2005) implies that there is a unique choice of $m_{t+1}(< m_{t+1}{}^*)$ in any equilibrium. That is, F_t must be degenerate at $M_{t+1} = M$.

By (10), the first-order condition with respect to m_{t+1} is

$$-\phi_t + \beta[v'_{t+1}(m_{t+1}) + \phi_{t+1}] = 0. \tag{11}$$

By (9),

$$v'_{t+1}(m_{t+1}) = \alpha\sigma[u'(q_{t+1}(m_{t+1})]q'_{t+1}(m_{t+1}) - \phi_{t+1} d'_{t+1}(m_{t+1}). \tag{12}$$

Inserting $\phi_t = \dfrac{Z(q_t)}{m_t}, q_t'(m_t) = \dfrac{\phi_t}{Z'(q_t)}$ and (12) into the first-order condition (11) and evaluate (11) at $m_{t+1} = M$, we have

$$Z(q_t) = \beta\, Z(q_{t+1})[\alpha\sigma \frac{u'(q_{t+1})}{Z'(q_{t+1})} + 1 - \alpha\sigma]. \tag{13}$$

An equilibrium of this economy can be defined as a value function $V_t(m)$ satisfying Bellman's equation (2), a solution to the bargaining problem given by $d = m$ and $q = \hat{q}(m)$, a bounded path for $\phi > 0$ such that (11) holds with $m = M$ and a balanced government budget. The above manipulations reduce the equilibrium conditions to (13), which is a simple difference equation in q_t .

4. Changes in the Money Supply

Now, suppose that the money supply grows over time through lump-sum transfers in the centralized market. That is, $M_{t+1} = (1+\tau)M_t$. Then (13) becomes

$$\frac{Z(q_t)}{M_t} = \beta \frac{Z(q_{t+1})}{M_{t+1}}[\alpha\sigma \frac{u'(q_{t+1})}{Z'(q_{t+1})} + 1 - \alpha\sigma]. \tag{14}$$

In the steady state, (14) becomes

$$\frac{u'(q)}{Z'(q)} = 1 + \frac{1+\tau-\beta}{\alpha\sigma\beta}. \tag{15}$$

At the Friedman rule, $\tau = \beta - 1$, so $u'(q) = Z'(q)$. By (7),

$$Z'(q) = \frac{u'(c'-s')[\theta\, u' + (1-\theta)(c'-s')] + \theta(1-\theta)(u-(c-s))[u'(c''-s'') - (c'-s')u'']}{[\theta\, u' + (1-\theta)(c'-s')]^2} \tag{16}$$

If $\theta = 1$, i.e. if the buyer has all the bargaining power, at the Friedman rule, the equilibrium allocation is efficient even if there are no subsidies, that is, $u'(q) = c'(q)$. This is the result obtained in Lagos and Wright (2005). If $\theta < 1$, i.e. if the buyer does not have all the bargaining power, at the Friedman rule, the equilibrium allocation is efficient $(u'(q) = c'(q) - s'(q))$ when

$$u'(c''-s'') - (c'-s')u'' = 0. \tag{17}$$

This can be achieved by providing appropriate subsidies.

5. Optimal Rate of Subsidy

To characterize the optimal subsidy, we need to specify the functional forms of the utility function, the cost function, and the subsidy function. Following Lagos and Wright (2005), we assume that $c(q) = q$ and $u(q) = \frac{q^{1-\eta}}{1-\eta}$ [2], where $\eta > 0$.

We propose three forms of subsidy function: $s_1(q_1) = \kappa_1 q_1$, $s_2(q_2) = 2\kappa_2 q_2^{1/2}$, $s_3(q_3) = \frac{1}{2}\kappa_3 q_3^{2}$, which correspond to that the subsidy function is linear, concave, and convex in q. The efficient production and the corresponding rate of subsidy can be jointly derived from equation (17) and the condition for efficient allocation, $u'(q) = c'(q) - s'(q)$. We impose the following reasonable conditions on the optimal rate of subsidy: (i) $\kappa_1 > 0$, $\kappa_2 > 0$, $\kappa_3 > 0$. (ii) $\kappa_1 < 1$, $\kappa_2 < \frac{1}{2}q_2^{1/2}$, $\kappa_3 < \frac{2}{q_3}$, which means that the amount of subsidy is less than the amount of production. We find that only the third subsidy function can satisfy all the conditions.

Specifically, for the first subsidy function, $s_1(q_1) = \kappa_1 q_1$, equation (17) becomes

$$(1-\kappa_1)\eta q_1^{-\eta-1} = 0,$$

which gives us $\kappa_1 = 1$, violating condition (ii).

For the second subsidy function, $s_2(q_2) = 2\kappa_2 q_2^{1/2}$, equation (17) becomes

$$\kappa_2(\eta - \frac{1}{2})q_2^{-1/2} - \eta = 0 \tag{18}$$

Efficient allocation implies

$$q_2^{-\eta} = 1 - \kappa_2 q_2^{-1/2} \tag{19}$$

Combining equations (18) and (19), we obtain the efficient production

$$q_2{*} = (1 - \frac{\eta}{\eta - \frac{1}{2}})^{-\frac{1}{\eta}}$$

and the optimal rate of subsidy

$$\kappa_2{*} = -2\eta(\frac{1}{1-2\eta})^{1-\frac{1}{2\eta}}$$

One can check that condition (i) is satisfied if $\eta > \frac{1}{2}$. Assuming $\eta > \frac{1}{2}$, condition (ii) implies $\eta < -\frac{1}{2}$, which is a contradiction and it violates the assumption $\eta > 0$.

For the third subsidy function, $s_3(q_3) = \frac{1}{2}\kappa_3 q_3^{\ 2}$, equation (17) becomes

$$-\kappa_3 q_3^{\ -\eta} + (1-\kappa_3 q_3)\,\eta\, q_3^{\ -\eta-1} = 0 \qquad (20)$$

Efficient allocation implies

$$q_3^{\ -\eta} = 1 - \kappa\, q_3 \qquad (21)$$

Combining (20) and (21), we obtain the efficient production

$$q_3{*} = (1+\eta)^{\frac{1}{\eta}} \qquad (22)$$

and the optimal rate of subsidy

$$\kappa_3{*} = \eta\,(\frac{1}{1+\eta})^{\frac{1}{\eta}+1} \qquad (23)$$

One can check that conditions (i) and (ii) are satisfied if $0 < \eta < 1$.
The optimal rate of subsidy depends on the relative risk aversion coefficient η. Since

$$\frac{d \log \kappa_3{*}}{d\eta} = -\frac{1}{\eta^2}\log\frac{1}{1+\eta} > 0$$

The more risk averse the buyer is, the higher is required the rate of subsidy. Thus, for a government that wishes to implement an efficient outcome, it turns out that the third subsidy function, $s_3(q_3) = \frac{1}{2}\kappa_3 q_3^{\ 2}$, leads to the desired result.

6. Conclusion

In reality, sellers have partial or full bargaining power. This note attempts to demonstrate one way that the Pareto efficient allocation can be restored at the Friedman rule by curing the inefficiency in the bargaining process. In principle, the same result can be obtained by taxing buyers. However, in an environment where agents meet randomly, taxes may be hard to implement. In Lagos and Wright (2005), buyers' partial bargaining power is associated with large welfare cost of inflation. The implication of this note is that if we add other tax/subsidy instruments in the economy, the welfare cost of inflation may be reduced.

References

[1] Aruoba, S.B. and S. Chugh (2010) Optimal fiscal and monetary policy when money is essential, Journal of Economic Theory, 145(5), 1618 – 1647.

[2] Correia, I, J. P. Nicolini and P. Teles (2004) Optimal Fiscal and Monetary Policy: Equivalence Results, Working Paper

[3] Diamond, P. (1982) Aggregate demand management in search equilibrium. Journal of Political Economy, 90, 881－894.

[4] Friedman, M. (1969) The optimum quantity of money. In: Friedman, Milton (ed.), The Optimum Quantity of Money and Other Essays, pp. 1 – 50. Chicago, Aldine Transaction.

[5] Gomis-Porqueras, P. and A. Peralta-Alva (2010) Optimal monetary and fiscal policies in a search theoretic model of monetary exchange, European Economic Review, 54, 331-344.

[6] Kocherlakota, N. (2005) Optimal monetary policy: what we know and what we don’t know. International Economic Review, 46, 715－729.

[7] Lagos, R. and G. Rocheteau (2005) Inflation, output, and welfare. International Economic Review, 46, 495 － 522.

[8] Lagos, R. and R.Wright (2005) A unified framework for monetary theory and policy analysis. Journal of Political Economy 113, 463-484

[9] Lagos, R..and R.Wright (2004) A unified framework for monetary theory and policy analysis. Staff Report No. 346 (September), Research. Department, Federal Reserve Bank of Minneapolis.

[10] Schmitt-Grohe, S. and M. Uribe (2010) The optimal rate of inflation, Handbook of Monetary Economics, Vol 3, North-Holland

Notes:

[1] Kocherlakota (2005) defines two strands of literature in monetary economics: the applied literature and the basic literature. The basic literature explicitly models the frictions that make money essential. The applied literature does not. Instead, money is introduced in an *ad hoc* fashion, as money-in-the utility function, cash-in-advance, shopping-time and transactions-cost technology models. In the applied literature, as summarized by Schmitt-Grohe and Uribe (2010), in models in which the only nominal friction stemming from a demand for fiat money for transaction purposes, the Friedman rule is optimal in the sense that it maximizes the welfare of the representative consumer by minimizing the opportunity cost of holding money. This result holds regardless of whether the government is assumed to finance its budget by lump-sum taxes or by distortionary income tax. The optimal monetary policy may deviate from the Friedman rule due to incomplete tax system or price stickiness. However, the Friedman rule will reemerge as the optimal monetary policy if the government has access to a sufficient number of tax instruments (Correia et al, 2004, Schmitt-Grohe and Uribe, 2010).

In the basic literature, generally speaking, the Friedman rule is optimal or constrained optimal (see Kocherlakota , 2005, for a review). For instance, in Lagos-Wright (2005), the Friedman rule is always optimal in the sense that a necessary condition for monetary equilibrium is that the growth rate of money supply should be at or above the Friedman rule. However, in an environment without a centralized Walrasian market, the mode of price determination may affect the nature of optimal monetary policy. As discussed in the second paragraph of this paper, the Friedman rule is not associated with the first best allocation when the buyer does not have all the bargaining power. In Aruoba and Chugh (2010), in the absence of proportional taxes and at the Friedman rule, the equilibrium under Nash bargaining can never achieve the social optimum due to the holdup problem present in the bargaining environment.

[2] In Lagos and Wright (2005), $u(q)=[(q+b)^{1-\eta}-b^{1-\eta}]/(1-\eta)$, so that $u(0)=0$. Since b can be a very small number, we set $u(q)=q^{1-\eta}/(1-\eta)$ for simplicity.