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A new axiomatization of the Shapley value under interval uncertainty

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Abstract

In the framework of interval games, we show that the Shapley value is the unique solution satisfying efficiency, symmetry and coalitional strategic equivalence.

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1 Introduction

In the theory of classical cooperative games, a characteristic function is a real-valued function defined over all the subsets of the set of players. Under the condition of "uncertainty", there are two extensions of the characteristic function in the literature. One is that the domain of the characteristic function is extended to allow "fuzzy" coalition.[1] The other is that the range of the characteristic function is extended to allow "fuzzy" value. The latter is the kind of situations that we want to discuss in this note.

Branzei, Dimitrov and Tijs [10] provided a game theoretic model to support decision making under interval uncertainty of coalition values, named interval games. The model of interval games fits all the situations where players consider cooperation and know with certainty only the lower and upper bounds of all potential revenues or costs generated via cooperation. Methods of interval arithmetic and analysis (cf. Moore [16]) have played a key role for new models of games based on interval uncertainty. In the meantime, many solution concepts have been developed. Related results may be found in Alparslan Gök, Branzei and Tijs [1, 2]; Alparslan Gök, Miquel and Tijs [3]; Branzei et al.[7, 8] , and so on.

The Shapley value [18] is a well-known solution concept in cooperative game theory. Two of the most appealing characterizations of the Shapley value are by Shapley [18] and by Young [19], respectively. Shapley proved that the Shapley value is the unique value satisfying four axioms, **efficiency**, **symmetry**, **dummy** and **additivity**. Replacing **dummy** and **additivity** by **marginality**, Young established an alternative axiomatic characterization of the Shapley value by three axioms, **efficiency**, **symmetry**, and **marginality**.

These mentioned above raise one question in the framework of interval games:

- whether the two axiomatic results of the Shapley value could be described in the framework of interval games.[2]

[1] Aubin [4, 5] first suggested to allow the players to choose any level of participation in a coalition, and he named "fuzzy game". Fuzzy games have proved to be suitable for modeling cooperative behavior of players in economic situations [6, 9] and political situations [11, 15] in which some players do not fully participate in a coalition but only to a certain extent. For example, in a class of production games, partial participation in a coalition means to offer a part of the resources while full participation means to offer all the resources.

[2] The interval Shapley value was introduced by Alparslan Gök, Branzei and Tijs

This note is aimed at answering the question. Instead of giving direct proofs, we establish a new axiomatization of the Shapley value by means of **efficiency, symmetry**, and **coalitional strategic equivalence**. Also, we provide two interesting logic relations between axioms:(1) **dummy** and **additivity** together imply **coalitional strategic equivalence** and (2) **marginality** implies **coalitional strategic equivalence**. Hence our result directly implies Shapley's result and Young's result.

2 Preliminaries

We follow the notation and terminology of Alparslan Gök, Branzei and Tijs [2]. Let $N = \{1, 2, \cdots, n\}$ be the set of players. $S \subseteq N$ is a coalition. The cardinality of S is denoted by $|S|$. Let $I(\mathbb{R})$ be the set of all nonempty and compact intervals in $\mathbb{R}$. A cooperative interval game is a pair (N, w), where $w : 2^N \to I(\mathbb{R})$ is a characteristic function such that $w(\emptyset) = [0, 0]$. For each $S \in 2^N$, the worth interval $w(S)$ of the coalition S in the interval game (N, w) is of the form $[\underline{w}(S), \overline{w}(S)]$, where $\underline{w}(S)$ is the minimal reward which coalition S could receive on its own and $\overline{w}(S)$ is the maximal reward which coalition S could get. The family of all interval games with player set N is denoted by IG^N. We denote by $I(\mathbb{R})^N$ the set of all such interval payoff vectors.

Let $I, J \in I(\mathbb{R})$ with $I = [\underline{I}, \overline{I}]$, $J = [\underline{J}, \overline{J}]$, $|I| = \overline{I} - \underline{I}$ and $\alpha \geq 0$. Then,

$$I + J = [\underline{I} + \underline{J}, \overline{I} + \overline{J}]; \tag{2.1}$$

$$\alpha I = [\alpha \underline{I}, \alpha \overline{I}]. \tag{2.2}$$

By (2.1) and (2.2) we see that $I(\mathbb{R})$ has a cone structure.[3]

For $(N, w_1), (N, w_2) \in IG^N$ and $\alpha \geq 0$ we define $(N, w_1 + w_2)$ and $(N, \alpha w_1)$ by $(w_1 + w_2)(S) = w_1(S) + w_2(S)$ and $(\alpha w_1)(S) = \alpha w_1(S)$ for each $S \in 2^N$.

We define $I - J$, only if $|I| \geq |J|$, by $I - J = [\underline{I} - \underline{J}, \overline{I} - \overline{J}]$. Note that $\underline{I} - \underline{J} \leq \overline{I} - \overline{J}$.

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[2] in the context of interval games. This is a generalization of the Shapley value for TU games. Also, they proved that the Shapley value is the unique solution satisfying four axioms, **efficiency**, **symmetry**, **dummy** and **additivity**.

[3]Let $I = [\underline{I}, \overline{I}] \in I(\mathbb{R})$, we call $\underline{I}$ and $\overline{I}$ are the lower bound and upper bound, respectively. Equation (2.1) guarantees that the lower (upper) bound satisfies additivity. Equation (2.2) guarantees that the lower (upper) bound satisfies scale invariance.

The model of interval cooperative games is an extension of the model of classical TU-games. We recall that a classical TU-game[4] $< N, v >$ is defined by $v : 2^N \to R$ and $v(\emptyset) = 0$. The unanimity game based on S, $u_S : 2^N \to R$ is defined by for all $T \subseteq N$,

$$u_S(T) = \begin{cases} 1 & , S \subseteq T \\ 0 & , \text{otherwise.} \end{cases}$$

A TU-game $< N, v >$ is monotonic if $v(S) \leq v(T)$ for all $S, T \in 2^N$ with $S \subseteq T$. We call an interval game (N, w) size monotonic if its length TU game $< N, |w| >$ is monotonic, where $|w|(S) = \overline{w}(S) - \underline{w}(S)$ for all $S \subseteq N$. We denote by $SMIG^N$ the class of size monotonic interval games with player set N.

Let $S \in 2^N \backslash \{\emptyset\}$, $I \in I(\mathbb{R})$ and let u_S be the unanimity game based on S. The cooperative interval game (N, Iu_S) is defined by $(Iu_S)(T) = u_S(T)I$ for all $T \subseteq N$. We denote by KIG^N the additive cone generated by the set

$$K = \{I_S u_S | S \in 2^N \backslash \{\emptyset\}\}$$

where $I_S \in I(\mathbb{R})$. That is, each element in KIG^N is a finite sum of elements of K. We notice that $KIG^N \subseteq SMIG^N$.

3 Interval Shapley value and Axioms

In the sequel, we focus on the set of games, KIG^N. A **solution** on KIG^N is a map ϕ assigning to each interval game $(N, w) \in KIG^N$ an element $\phi(N, w) \in I(\mathbb{R})^N$. The interval Shapley value[5] $\Phi : KIG^N \to I(\mathbb{R})^N$ is defined by

$$\Phi_i(N, w) := \sum_{S: i \in S} \frac{(|S| - 1)!|N \setminus S|!}{|N|!} \{w(S) - w(S \setminus \{i\})\},$$

for each $i \in N$ and for each $(N, w) \in KIG^N$.

Remark 3.1 *An alternative definition of the interval Shapley value for an interval game (N, w) is as follows.*

$$\Phi_i(N, w) = \sum_{\substack{S \subseteq N \\ i \in S}} \frac{I_S(w)}{|S|} = \sum_{\substack{S \subseteq N \\ i \in S}} d_S(w),$$

[4]We use $< N, v >$ and (N, v) to denote a TU game and an interval game, respectively.

[5]Alparslan Gök, Branzei and Tijs [2] defined the interval Shapley value on $SMIG^N$.

where $(N, w) \in KIG^N$ *with* $w = \sum_{S \in 2^N \setminus \emptyset} I_S(w) u_S = \sum_{S \in 2^N \setminus \emptyset} |S| d_S(w) u_S$.

From now on, if it is no ambiguous, we use w instead of (N, w).
We need the following axioms.

Efficiency (Eff): $\sum_{i \in N} \phi_i(w) = w(N)$ for all $w \in KIG^N$.

Let $w \in KIG^N$ and $i, j \in N$. i and j are called *symmetric players* in w, if $w(S \cup \{i\}) - w(S) = w(S \cup \{j\}) - w(S)$, for all $S \subseteq N \setminus \{i, j\}$.

Symmetry (Sym): if $i, j \in N$ are symmetric players in w, $\phi_i(w) = \phi_j(w)$ for all $w \in KIG^N$.

Let $w \in KIG^N$ and $i \in N$. Then, i is called a *dummy player* in w if $w(S \cup \{i\}) = w(S) + w(\{i\})$, for all $S \subseteq N \setminus \{i\}$.

Dummy player property (Dpp): if i is a dummy player in w, $\phi_i(w) = w(\{i\})$ for all $w \in KIG^N$.

Additivity (Add): $\phi(v + w) = \phi(v) + \phi(w)$ for all $v, w \in KIG^N$.

Let $w \in KIG^N$ and $i \in N$, let $\triangle_i w(S)$ be defined by

$$\triangle_i w(S) = \begin{cases} w(S) - w(S \setminus \{i\}) & \text{if } i \in S, \\ w(S \cup \{i\}) - w(S) & \text{if } i \notin S. \end{cases}$$

$\triangle_i w(S)$ represents the marginal contribution interval of i to S, and $\triangle_i w$ is the marginal contribution interval vector of i.

Marginality (Mar): if $\triangle_i v = \triangle_i w$, then $\phi_i(v) = \phi_i(w)$, where $v, w \in KIG^N$ and $i \in N$.

Coalitional strategic equivalence (Cse): for all $T \subseteq N$ such that $T \neq \emptyset$, and for all $I \in I(\mathbb{R})$, if $v = w + I u_T$, then $\phi_i(v) = \phi_i(w)$ for all $i \in N \setminus T$, where $v, w \in KIG^N$.

Strategic equivalence, introduced by von Neumann and Morgenstern [17], requires that adding a constant to the worths of all coalitions containing a given player i does not affect the payoffs of other players. Chun [14] introduced a variant version of strategic equivalence in TU games,

coalitional strategic equivalence.[6] It requires that adding a constant to the worths of all coalitions containing a given coalition T does not affect the payoffs of players that do not belong to T. Here we provide a version of **coalitional strategic equivalence** in the framework of interval games.

Lemma 3.1 *The interval Shapley value satisfies* **Eff, Sym, Dpp, Add, Mar** *and* **Cse**.

Proof. The proof is straightforward, we omit it. ∎

Two logical relations between axioms are as follows.[7]

Lemma 3.2 **Dpp** *and* **Add** *together imply* **Cse**.

Proof. Let $v, w \in KIG^N$ be two games satisfying the hypotheses of **Cse**, i.e., $v = w + Iu_T$ for some $T \subseteq N$ and for some $I \in I(\mathbb{R})$. For all $i \in N \setminus T$ and for all S with $i \notin S$, $Iu_T(S \cup \{i\}) - Iu_T(S) = Iu_T(\{i\}) = [0, 0]$, so that, by **Dpp**, we have

$$\phi_i(Iu_T) = Iu_T(\{i\}) = [0, 0] \text{ for all } i \in N \setminus T. \tag{3.1}$$

On the other hand, by **Add**, we have

$$\phi_i(v) = \phi_i(w) + \phi_i(Iu_T) \text{ for all } i \in N. \tag{3.2}$$

From Equations (3.1) and (3.2), we have, for all $i \in N \setminus T$,

$$\phi_i(v) = \phi_i(w) + \phi_i(Iu_T) = \phi_i(w).$$

∎

Remark 3.2 *The converse of Lemma 3.2 is not true in general. Let ϕ be the solution on KIG^N defined by for all $v \in KIG^N$,*

$$\phi_i(v) = \begin{cases} [1,1] & , \textit{ if } i = 1 \\ [0,0] & , \textit{ otherwise,} \end{cases}$$

where $i \in N$.
Then ϕ satisfies **Cse** *but it does not satisfy either* **Dpp** *or* **Add**.

[6] In the framework of TU games, Chun [14] provided another axiomatization of the Shapley value by means of **efficiency, triviality, fair ranking**, and **coalitional strategic equivalence**.

[7] Chun [14] established the two interesting results in the framework of TU games.

Lemma 3.3 **Mar** *implies* **Cse***.*

Proof. Let $v, w \in KIG^N$ be two games satisfying the hypotheses of **Cse**, i.e., $v = w + Iu_T$ for some $T \subseteq N$ and for some $I \in I(\mathbb{R})$. It is easy to see that for all $i \in N \setminus T$, $\triangle_i v = \triangle_i w$. Hence, by **Mar**, we have $\phi_i(v) = \phi_i(w)$ for all $i \in N \setminus T$. ∎

Remark 3.3 *In the framework of TU games it was recently proved the equivalence between* **Cse** *and* **Mar** *(Casajus [12], Casajus and Huettner [13]). This equivalence is still true here. The proof is similar to that of Proposition (Casajus and Huettner [13]) and, hence, it is left to reader.*

4 Main Result

In this section, we present an axiomatic characterization of the interval Shapley value by means of **Eff, Sym**, and **Cse**.

Theorem 4.1 *The interval Shapley value is the unique solution satisfying* **Eff, Sym***, and* **Cse***.*

Proof. We only need to show the "uniqueness". Let ϕ be a solution satisfying **Eff, Sym** and **Cse**. We first exploit the fact noted in Remark 3.1 that every interval game $v \in KIG^N$ can be expressed as

$$v = \sum_{\emptyset \neq T \subseteq N} I_T u_T. \tag{4.1}$$

The interval Shapley value can be expressed as $\Phi_i(v) = \sum_{\emptyset \neq T \subseteq N} \Phi_i(I_T u_T) = \sum_{T: i \in T} \frac{I_T}{|T|}$.

Define the index B of v to be the minimum number of non-zero terms in expression for v of Equation (4.1). The proof continues by induction on B.

If $B = 0$, then v is the trivial interval game. That is, $v(S) = [0, 0]$ for all $S \subseteq N$. By **Eff** and **Sym**, $\phi_i(v) = [0, 0]$ for all $i \in N$.

Assume now that $\phi(v)$ is the interval Shapley value whenever the index of v is at most B, and let v have index $B + 1$ with expression

$$v = \sum_{k=1}^{B+1} I_{T_k} u_{T_k}, \text{ all } I_{T_k} \neq [0, 0].$$

Define the game $v_{-p} = \sum_{k=1:k\neq p}^{B+1} I_{T_k} u_{T_k}$ for all $p = 1, 2, \cdots, B+1$. Then for all $p = 1, 2, \cdots, B+1$,

$$v = v_{-p} + I_{T_p} u_{T_p}.$$

By applying **Cse** $(B+1)$ times to all $p = 1, 2, \cdots, B+1$, we have that for all $p = 1, 2, \cdots, B+1$,

$$\phi_i(v) = \phi_i(v_{-p}) \text{ for all } i \in N \setminus T_p. \tag{4.2}$$

Since for all $p = 1, 2, \cdots, B+1$, the index of v_{-p} is B, by the hypothesis of induction, we derive that for all $i \in N \setminus T_p$,

$$\phi_i(v_{-p}) = \Phi_i(v_{-p}). \tag{4.3}$$

For all $p = 1, 2, \cdots, B+1$, combining the fact "$\Phi_i(v) = \Phi_i(v_{-p})$ for all $i \in N \setminus T_p$" with Equations (4.2) and (4.3), we derive that for all $i \in N \setminus T_p$,

$$\phi_i(v) = \phi_i(v_{-p}) = \Phi_i(v_{-p}) = \Phi_i(v).$$

Hence for all $i \notin \bigcap_{k=1}^{B+1} T_k$,

$$\phi_i(v) = \Phi_i(v) = \sum_{k=i\in T_k} \frac{I_{T_k}}{|T_k|}.$$

It remains to show that $\phi_i(v) = \Phi_i(v)$ for all $i \in \bigcap_{k=1}^{B+1} T_k$. By **Sym**, $\phi_i(v)$ is a same interval J for all members of $\bigcap_{k=1}^{B+1} T_k$; likewise $\Phi_i(v)$ is some interval J' for all members of $\bigcap_{k=1}^{B+1} T_k$. Since both solutions sum to $v(N)$ and are equal for all $i \notin \bigcap_{k=1}^{B+1} T_k$, it follows that $J = J'$. ∎

By Lemmas 3.1-3.3, Theorem 4.1 directly implies the following two theorems.

Theorem 4.2 *(Alparslan Gök, Branzei and Tijs [2]) The interval Shapley value is the unique solution satisfying* **Eff, Sym, Dpp** *and* **Add**.

Theorem 4.3 *The interval Shapley value is the unique solution satisfying* **Eff, Sym**, *and* **Mar**.

5 Independence of the Axioms

The following examples show that the independence of axioms in Theorem 4.1.

Example 5.1 *Let ϕ^1 be the solution on KIG^N defined by for all $v \in KIG^N$ and for all $i \in N$,*

$$\phi_i^1(v) = [0, 0].$$

Then ϕ^1 satisfies **Sym** *and* **Cse** *but it violates* **Eff**.

Example 5.2 *We first recall that every interval game $v \in KIG^N$ can be expressed as $v = \sum_{\emptyset \neq T \subseteq N} I_T u_T$ and the interval Shapley value can be expressed as $\Phi_i(v) = \sum_{T: i \in T} \frac{I_T}{|T|}$. Let ϕ^2 be a "weighted" interval Shapley value defined by for all $i \in N$,*

$$\phi_i^2(v) = \sum_{T: i \in T} \frac{\lambda_i}{\sum_{j \in T} \lambda_j} I_T,$$

where $\lambda_i = \frac{i}{\sum_{j \in N} j}$.
Then ϕ^2 satisfies **Eff** *and* **Cse** *but it violates* **Sym**.

Example 5.3 *Let ϕ^3 be the solution on KIG^N defined by for all $v \in KIG^N$ and for all $i \in N$,*

$$\phi_i^3(v) = \frac{v(N)}{n}.$$

Then ϕ^3 satisfies **Eff** *and* **Sym** *but it violates* **Cse**.

6 Discussion

In general, interval games can be mathematically seen as special instances as vector-valued games with two ordered components. More precisely, in vector-valued games the worth of a coalition is given by a vector of real numbers rather than by a real number. The components of such vectors are sharp values for the worth of all coalitions from the point of view of a finite number of criteria under consideration. Since any interval of real numbers can be regarded as a point in $\mathbb{R}^2$, an interval game can be seen as a special vector-valued game with two components, where the

first component should not be greater than the second component. The referee pointed out an interesting question:
From "real numbers" to "intervals of real numbers" only makes a difference in dimension. Intuitively, it might be that many results from classical theory (TU games) could just be imported to this new theory (interval games) by replicating the proof. The referee's question is: whether this new theory can yield new and surprising results that have no counterparts in the classical theory? We do not have too many ideas. We only know that some results in TU games does not apply to interval games because the subtraction of two intervals is limited. For example, the core satisfies the translation invariance property, but the interval core[8] violates the translation invariance property. We first introduce some well-known definitions and notation in order to illustrate this fact.

The interval imputation set $I(w)$ of the interval game w, is defined by

$$I(w) = \{(I_i)_{i\in N} \mid \sum_{i\in N} I_i = w(N), \underline{I}_i \geq \underline{w}(\{i\}), \overline{I}_i \geq \overline{w}(\{i\}) \ \forall i \in N\},$$

and the interval core $C(w)$ of the interval game w, is defined by

$$C(w) = \{I \in I(w) \mid \sum_{i\in S} \underline{I}_i \geq \underline{w}(S), \sum_{i\in S} \overline{I}_i \geq \overline{w}(S) \ \forall S \in 2^N \setminus \{\emptyset\}\}.$$

The following example illustrates that the interval core does not satisfy the translation invariance property. That is, there exist an interval game w and an interval payoff vector $a \in I(\mathbb{R})^N$ such that $C(w+a) \neq C(w)+a$, where $(w+a)(S) = w(S) + \sum_{i\in S} a_i$ for all $S \in 2^N \setminus \{\emptyset\}$, and $C(w)+a = \{b+a \mid b \in C(w)\}$.

Let $N = \{1,2\}$ and $w(\{1\}) = [0,0]$, $w(\{2\}) = [-1,0]$, $w(\{1,2\}) = [0,0]$, and $a = ([1,3],[1,1])$. Then $(w+a)(\{1\}) = [1,3]$, $(w+a)(\{2\}) = [0,1]$, $(w+a)(\{1,2\}) = [2,4]$. Let $I = ([\underline{I}_1, \overline{I}_1], [\underline{I}_2, \overline{I}_2]) \in C(w)$. By the definition of $C(w)$, $\underline{I}_1 \geq 0, \overline{I}_1 \geq 0, \underline{I}_2 \geq -1, \overline{I}_2 \geq 0$, $\underline{I}_1 + \underline{I}_2 = 0$, and $\overline{I}_1 + \overline{I}_2 = 0$. Since $\underline{I}_1 \leq \overline{I}_1$, this forced $\underline{I}_1 = \overline{I}_1 = 0$. So, $C(w) = \{([0,0],[0,0])\}$. Hence $C(w)+a = \{a\} = \{([1,3],[1,1])\}$. On the other hand, it is also easy to see that $x = ([2,3],[0,1]) \in C(w+a)$. But $x \notin C(w)+a$. Hence $C(w+a) \neq C(w)+a$.

[8]The interval core was introduced by Alparslan Gök, Branzei and Tijs [1].

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