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On the instability of competitive equilibrium: a further example

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Abstract

It is shown that a Walrasian price adjustment process fails to converge to an equilibrium in an exchange economy with three consumers and three commodities, where each consumer has a quasilinear utility function, desires only two commodities, and demands positive amounts of both commodities. The instability is due to weak substitution effects in addition to asymmetrical income effects.

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1. Introduction

The purpose of this paper is to point out a hitherto unnoticed implication of Scarf's (1960) seminal work on three commodity economies with a unique unstable equilibrium. The instability result of Scarf was illustrated under the CES family of preferences [see also Hirota (1981, 1985)],[1] but it is also valid under a quasilinear preference, provided that all consumers choose interior solutions — to our knowledge, except for Gale's (1963) example (Giffen's case), similar results have not been obtained with other types of preferences. Although this result itself is not surprising in light of the Sonnenschein-Mantel-Debreu results which imply that any continuous function satisfying Walras Law and Homogeneity can be realized as the excess demand functions of some exchange economy, the example presented here gives explicit conditions under which a Walrasian price adjustment process fails to converge to an equilibrium. The meaning of the example is thus twofold: (i) to raise a note of caution when, for example, performing comparative statics with quasilinear preferences and (ii) to provide an economic foundation for the instability of competitive equilibrium. That is, quasilinear preferences, widely used in economics, must be treated carefully in equilibrium analysis (unlike Cobb-Douglas preferences which exhibit gross substitutability). The finding might have practical value, since Anderson et. al (2004) observe that the average transaction prices in double auction experiments follow the path predicted by the Scarf and Hirota models. In addition, an interpretation of the example is that it shows instability in an exchange between three kinds of money [for an exchange between two kinds of money, see Shapley and Shubik (1977) and also Bergstrom et. al (2009)].

2. Model and Results

Let us consider a class of three commodity economies. The excess demand functions $Z_i(p_1, p_2, p_3) : \mathbb{R}^3_{++} \to \mathbb{R}, i = 1, 2, 3$ are assumed to satisfy the following: For all $(p_1, p_2, p_3) \in \mathbb{R}^3_{++}$,

(A1) $\sum_i^3 p_i Z_i = 0$ (Walras Law),

(A2) $\forall i$, $Z_i(\alpha p_1, \alpha p_2, \alpha p_3) = Z_i(p_1, p_2, p_3)$ for any $\alpha > 0$ (Homogeneity),

(A3) Each Z_i is continuously differentiable (Differentiability),

(A4) $Z_2(p_1, p_2, p_3) = Z_1(p_2, p_3, p_1)$ and $Z_3(p_1, p_2, p_3) = Z_2(p_2, p_3, p_1)$ (Circularity),

(A5) $\partial Z_1/\partial p_2 < 0$ and $\partial Z_1/\partial p_3 > 0$.

[1] To be exact, Scarf offered two examples: one with Leontief type utility functions and the other with CES utility functions.

Then, Lemmas 1–3 in Scarf (1960) state that the excess demand functions Z_i have an equilibrium at $(1, 1, 1)$ and the equilibrium is unique up to a positive scalar multiple.

The price adjustment process which we consider is as follows:

$$\dot{p}_i = Z_i(p_1, p_2, p_3), \quad p_i^0 > 0 \quad i = 1, 2, \quad \text{with } p_3 \equiv 1, \tag{1}$$

where $\dot{p}_i$ denotes the time derivative of $p_i(t)$ and $p_i^0 = p_i(0)$. The Jacobian matrix of the right hand side at the equilibrium $(1, 1)$ can be expressed as

$$\begin{bmatrix} C & A \\ B & C \end{bmatrix}, \tag{2}$$

whose eigenvalues are: $C \pm \sqrt{AB}$. Since the product AB is negative, the stability depends on the sign of C; if C is positive, then the equilibrium is locally unstable. This result is parallel to Lemma 4 of Scarf (1960) for the price adjustment process on the sphere.

We now consider an exchange economy with commodity space $\mathbb{R}^3$ and three consumers. Suppose that the utility functions and initial endowments are:

$$\begin{aligned}
&\text{consumer 1}: \quad \{\ u^1(x_1, x_2, x_3) = v(x_1, x_2),\ \omega^1 = (b_1, 0, 0)\ \},\\
&\text{consumer 2}: \quad \{\ u^2(x_1, x_2, x_3) = v(x_2, x_3),\ \omega^2 = (0, b_2, 0)\ \},\\
&\text{consumer 3}: \quad \{\ u^3(x_1, x_2, x_3) = v(x_3, x_1),\ \omega^3 = (0, 0, b_3)\ \},
\end{aligned}$$

where v is the quasilinear utility function on $\mathbb{R} \times \mathbb{R}_{++}$ defined by $v(y_1, y_2) = y_1 - (1/a)y_2^{-a}$, $a > 0$, and $b_i > 0$, $i = 1, 2, 3$. For our purpose, we focus on interior solutions to avoid boundary issues. Consumer i demands positive amounts of both commodities:

$$y_1 = b_i - (p_{i+1}/p_i)^{a/(1+a)},\ y_2 = (p_i/p_{i+1})^{1/(1+a)} \quad \text{if} \quad p_{i+1}/p_i < b_i^{(1+a)/a}, \tag{3}$$

where $i + 1 = 1$ for $i = 3$. The excess demand functions are then given by

$$\begin{aligned}
Z_1(p_1, p_2, p_3) &= (p_3/p_1)^{1/(1+a)} - (p_2/p_1)^{a/(1+a)},\\
Z_2(p_1, p_2, p_3) &= (p_1/p_2)^{1/(1+a)} - (p_3/p_2)^{a/(1+a)},\\
Z_3(p_1, p_2, p_3) &= (p_2/p_3)^{1/(1+a)} - (p_1/p_3)^{a/(1+a)},
\end{aligned} \tag{4}$$

which clearly satisfy (A1)–(A5).

Consider the process (1) with the functions (4). In what follows, we always restrict ourselves to the open region E in the (p_1, p_2)-plane with $p_3 = 1$ containing the equilibrium $(1, 1)$ such that the interiority conditions of (3) are all satisfied, whose existence is guaranteed for $b_i > 1$, $i = 1, 2, 3$; we consider the process on $E = \{(p_1, p_2) \in \mathbb{R}^2_{++} | \forall i, p_{i+1}/p_i < b_i^{(1+a)/a}, p_3 = 1, b_i > 1, \ (i+1 = 1 \text{ for } i = 3)\}$ with $(p_1^0, p_2^0) \in E$. Then, C in the matrix (2) is: $-(1-a)/(1+a)$, which means that the unique equilibrium $(1, 1)$ is locally unstable for $a > 1$;[2] this instability occurs when each excess demand is an increasing function of its own price at the equilibrium. Let us note that (i) none of the commodities is inferior and (ii) the instability is due to weak substitution effects in addition to asymmetrical income effects.[3] In fact, writing the C as the sum of substitution and income terms, one can verify that the substitution terms tend to zero as a increases. For disequilibrium dynamics, we have the following results:

Proposition 1. For $a = 1$, the process has a continuous family of closed orbits around $(1, 1)$.

Proof. [4] Let $a = 1$. Let us introduce the function on $\mathbb{R}^2_{++}$ defined by $H(p_1, p_2) = h(p_1) + h(p_2)$ where $h(y) = 1/6 + (1/2)y^2 - (2/3)y^{3/2}$. We find that along any solution of the process, $\dot{H} = 0$; consequently H is a first integral of the process. Notice that $dh/dy \gtreqqless 0 \Leftrightarrow y \gtreqqless 1$. This implies that H has a global minimum at $(1, 1)$, $H(1, 1) = 0$, and is not constant on any open set. Then, we can take a region $R \subset E$ bounded by a closed level curve of H, i. e., $R = \{(p_1, p_2) \in \mathbb{R}^2_{++} | H(p_1, p_2) \leqq k, 0 < k < 1/6\}$, and conclude, by Theorem 3 in Hirsch and Smale (1974, p. 252), that there is no limit cycle in R. Moreover, R is compact and invariant, and hence the ω-limit set of any initial point in R is nonempty and compact. Thus, by the Poincaré-Bendixson Theorem [c.f. Hirsch and Smale (1974, p. 248)], if the ω-limit set does not contain an equilibrium, i. e., $(1, 1)$, then it is a closed orbit. Consider an arbitrary initial point $(p_1^0, p_2^0) \in R \setminus \{(1, 1)\}$. Since $H(p_1^0, p_2^0) > 0$ and H is constant along every solution, the equilibrium $(1, 1)$ cannot be in the ω-limit set of the point (p_1^0, p_2^0); therefore the ω-limit set must be a closed orbit. This, together with the nonexistence of limit cycles, finishes the proof. □

Proposition 2. For $a > 1$, the process has no closed orbit.

[2] It is known that if there is a unique, locally completely unstable equilibrium (all eigenvalues have positive real values) in a regular economy, then the number of commodities is odd [Dierker (1974, Sec. 11)]. However, our example can not be seen as a direct application of the result, since we adopt the *numéraire* normalization and have the interiority conditions.

[3] We have stability if we neglect asymmetrical income effects. See Arrow and Hurwicz (1958) and Arrow and Hahn (1971, Sec. 12.5).

[4] This proof is inspired by Flaschel's (1984) proof of the closed-orbit structure of Goodwin's growth cycle model.

Proof. Let $a > 1$. Let us introduce the function on $\mathbb{R}^2_{++}$ defined by $W(p_1, p_2) = (p_1 p_2)^{a/(1+a)}$. We have: $(\partial/\partial p_1)(WZ_1) + (\partial/\partial p_2)(WZ_2) = -[(1-a)/(1+a)]$ $[(p_1^{1+a}/p_2^2)^{1/(1+a)} + (p_2^a/p_1^2)^{1/(1+a)}]$, which is positive. It then follows, by Dulac's Criterion [c.f. Andronov et. al (1966, p. 305)], that there can be no closed orbit. $\square$

We can also characterize the type of the equilibrium $(1, 1)$: $a < 1$ stable focus (i. e., solutions near it spiral toward it); $a = 1$ center (i. e., solutions near it are periodic); $a > 1$ unstable focus (i. e., solutions near it spiral away from it). Finally, it should be remarked that the price dynamics of our example is similar to that of the perturbed Scarf example in Mukherji (2007). The mechanisms of loss of stability are different, however. In the latter environment there are no substitution effects and the loss of stability occurs as the initial endowment of one commodity increases, whereas our instability arises as the parameter in the utility function increases and it is attributed to weak substitution effects as well as asymmetrical income effects.

References

Anderson, C. M., C. R. Plott, K.-I. Shimomura, and S. Granat (2004) "Global instability in experimental general equilibrium: The Scarf example" *Journal of Economic Theory* 115, 209–249.

Andronov, A. A., A. A. Vitt, and S. E. Khaikin (1966) *Theory of Oscillators*, (translated from the Russian by Immirzi, F.; the translation edited and abridged by Fishwick, W.) Oxford: Pergamon Press.

Arrow, K. J. and L. Hurwicz (1958) "On the stability of the competitive equilibrium, I" *Econometrica* 26, 522–552.

Arrow, K. J. and F. H. Hahn (1971) *General Competitive Analysis*, Amsterdam: North-Holland.

Bergstrom, T. C., K.-I. Shimomura, and T. Yamato (2009) "Simple economies with multiple equilibria" *The B.E. Journal of Theoretical Economics* 9, 1–29.

Dierker, E. (1974) *Topological Methods in Walrasian Economics*, Berlin: Springer-Verlag.

Flaschel, P. (1984) "Some stability properties of Goodwin's growth cycle: A critical elaboration" *Zeitschrift für Nationalökonomie* 44, 63–69.

Gale, D. (1963) "A note on global instability of competitive equilibrium" *Naval Research Logistics Quarterly* 10, 81–87.

Hirota, M. (1981) "On the stability of competitive equilibrium and the patterns of initial holdings: An example" *International Economic Review* 22, 461–467.

Hirota, M. (1985) "Global stability in a class of markets with three commodities and three consumers" *Journal of Economic Theory* 36, 186–192.

Hirsch, M. W. and S. Smale (1974) *Differential Equations, Dynamical Systems, and Linear Algebra*, New York: Academic Press.

Mukherji, A. (2007) "Global stability condition on the plane: A general law of demand" *Journal of Economic Theory* 134, 583–592.

Scarf, H. (1960) "Some examples of global instability of competitive equilibrium" *International Economic Review* 1, 157–172.

Shapley, L. S., and M. Shubik (1977) "An example of a trading economy with three competitive equilibria" *Journal of Political Economy* 85, 873–875.