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A subjective discounted utility model

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Abstract

Intertemporal choices involve a great heterogeneity among discount mechanisms. In order to catch such diversity, we introduce an axiomatic Subjective Discounted Utility (SDU) model based on separability assumption. The originality of the SDU model rests on the fact that decision makers discount subjective periods, namely decision weights can be described as standard discount functions of time perception. In particular, our model appears as a generalization of both exponential and hyperbolic approaches.

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1. Introduction

Intertemporal choices deal with tradeoffs between rewards that occur over time. An immediate consequence is that discounting, which expresses the relative weight of future periods compared with the present period, is the cornerstone of intertemporal decision. In this paper, we focus on models where separability is assumed namely rewards are independently discounted from each others. Such models will be called Discounted Utility (DU) models. The intertemporal reference model is the Exponential Discounted Utility (EDU) where decision weights follow geometrical decrease with time. The axiomatic foundations of the EDU model, built around the specific axiom of stationarity, were provided by Koopmans (1972) or Fishburn and Rubinstein (1982). For a few decades, a large body of empirical literature has shown that stationarity is systematically violated and that the way of discounting future periods varies considerably from one study to another. This has led to a growing diversity of discount mechanisms in theoretical models. In particular, alternative models named Hyperbolic Discounting Utility (HDU) as in Loewenstein and Prelec (1992) or as in Laibson (1997)[1] have attempted to catch the property of decreasing impatience which states that Decision Makers (DM) are the more sensitive to timing changes the closer these changes are to the present. More recently, other empirical studies even found increasing impatience as in Gigliotti and Sopher (2004) and such a behavior has been axiomatized in Bleichrodt Rohde and Wakker (2009). However before Albrecht and Weber (1995), there was no clear reason explaining the great diversity of elicited discount functions. Albrecht and Weber provided a decisive insight on the topic by introducing time perception into standard discount mechanism. By analogy to space perception, time perception may lead to deformation of future time intervals especially when they are very distant from the observation period. In this way the abundant variety of time perceptions may accommodate the great diversity in time discounting. Other fields of human sciences such as Psychology[2] have undertaken to implement the individual perception of time in order to explain intertemporal choices. In the model providing by Albrecht and Weber, time perception is applied to distant future (a month, a year or even a decade) contrary to the traditional use of time perception in Psychology which is mainly dedicated to the “specious” present (a few seconds from now). In particular, the two authors presented hyperbolic discounting as an exponential discounting of a Weber-Fechner time perception[3]. That is why it is quite paradoxical that time perception has raised so little interest from economists. A first explanation would be that the Weber-Fechner “law” (or other psychological laws) has been built for very short time intervals whereas most of economic studies have an interest for choices involving more delayed rewards. In this paper, time perception functions are defined for short term periods as well as for long term time intervals. We argue that the lack of axiomatic foundation for time perception is the major cause of the limited diffusion of time perception in discounting approaches. The main goal of this paper is to provide a sound axiomatic foundation of a separable representation of preferences where decision weights are defined as in Albrecht and Weber model. By analogy to the well known Subjective Expected Utility defined for uncertain choices, we call such a model Subjective Discounted Utility (SDU) to emphasize that discounting rests on time perception. The SDU model is compatible with any discount mechanism and both generalizes the EDU model and the HDU model. This paper is organized as follows. Section 2 gives notations and specifies the environment on which the SDU will be built. Section 3 provides

[1] Loewenstein and Prelec define their model as Generalized Hyperbolic Discounting whereas Laibson calls it Quasi Hyperbolic Discounting.
[2] In Psychology, for instance, an important approach for time perception is the Internal Clock Theory.
[3] Ernst Weber (1795-1878) and Gustav Fechner (1801-1887) pointed out a logarithmic time perception for which delayed time intervals are underestimated.

the main result and defines a clear relationship between the degree of impatience and the nature of time perception. In the end, section 4 concludes.

2. Intertemporal Preferences with Separability

In this section, the general framework needed for our main theorem is stated. For an index set $S = \{0, \dots, k, \dots, n\}$, a time sequence $t = \{t_0, \dots, t_k, \dots, t_n\}$ is a series of $n+1$ periods where period $t_k \in \mathcal{P}$ ($\mathcal{P}$ is a subset of $\mathbb{R}^+$ containing 0) and the set of time sequences is $T = \{t : S \to \mathcal{P}$ with $\forall k \in S, t_k \geq 0$ and $\forall k \in S - \{n\}, t_k < t_{k+1}\}$. In the same way, an outcome sequence $x = \{x_0, \dots, x_k, \dots, x_n\}$ is a series of $n+1$ outcomes where $x_k \in \mathcal{C}$ ($\mathcal{C}$ is a subset of $\mathbb{R}$) and the set of outcome sequences is $X = \{x : S \to \mathcal{C}\}$. The decision set is $\Delta = \mathrm{X} \times \mathrm{T}$ and gathers all the finite timed outcome sequences. As a consequence, a choice alternative in Δ is a collection of future dated rewards $(x, t) = \{x_0, \dots, x_k, \dots, x_n; t_0, \dots, t_k, \dots, t_n\} = \{(x_k; t_k)\}_{k \in S}$ where $(x_k; t_k)$ means that outcome x_k is received at period t_k. The goal of this section is to provide a DU representation for time preferences, that is the intertemporal utility function satisfies separability in the following sense $U(x,t) = \sum_{k \in S} \mu(t_k) u(x_k)$ (1). For simplicity, we introduce the following notations for common future dated rewards. For a given time sequence t, the notation $(z_i a, t)$ means that reward for period i of the outcome sequence (z, t) (with $i \in t$) is replaced with the reward a at the same period. Similarly the notation $\left(z_{i,j} a, b, t\right)$ means that rewards for periods i and j of the outcome sequence (z, t) (with $i, j \in t$) are respectively replaced with rewards a and b at the same periods and so on. A binary relation $\succsim$ is defined on $\Delta \times \Delta$. As usual, $\succ$ and $\sim$ are defined to be respectively the asymmetric and symmetric parts of $\succsim$. We introduce a set of axioms which are defined for all t in T and for all periods i, j, k, l in time sequence $(i < j; k < l)$, for all $x, y \in X$ and for all $a, b, c, d \in \mathcal{C}$.

A.1 Regularity

A regular binary relation $\succsim$ is a continuous monotonic weak order[4] and has neutral element x^* in X such that $\forall s, t \in T$ and for all subset A in S:

$$(x^*_A x, s) \sim (x^*_A x, t)$$

With a slight abuse of notation, we shall write the neutral element in X as follows: $x^* = (x^*, x^*, \dots, x^*)$ since x^* is neutral on time preferences if is neutral for each period. By simplification, we shall note finite timed outcome sequences which have non neutral rewards on period set A as follows $(x^*_A x, s) \equiv (x^*_A x)$.

A.2 Separability

First Order Separability (A.2.a)

$$\left(x_{i,i+1} a, b, t\right) \succsim \left(x_{i,i+1} c, d, t\right) \Longrightarrow \left(y_{i,i+1} a, b, t\right) \succsim \left(y_{i,i+1} c, d, t\right)$$

Second Order Separability (A.2.b)

$$\text{If} \quad \begin{matrix} (x^*{}_i a) \sim \left(x^*{}_j b\right) \\ (x^*{}_k a) \sim \left(x^*{}_l b\right) \end{matrix} \quad \text{then} \quad (x^*{}_i c) \sim \left(x^*{}_{i,j} a, b\right) \Longrightarrow (x^*{}_k c) \sim \left(x^*{}_{k,l} a, b\right)$$

[4] A weak order is a transitive and complete binary relation. We retain strict monotonicity ($a > b$ implies $(x_i a, t) \succ (x_i b, t)$). Continuity applies twice: for $(x, s) \succ (y, t)$, $\forall k \in S$, $\exists\, \delta_k > 0$ which is strictly preferential preserving such that $(x', s) \succ (y, t)$ with $x' = (x_0, \dots, x_{k-1}, x_k - \delta_k, x_{k+1}, \dots, x_n)$ and $(x, s') \succ (y, t)$ with $s' = (s_0, \dots, s_{k-1}, s_k + \delta_k, s_{k+1}, \dots, s_n)$.

Separability is the key axiom for the DU model. Separability raises simultaneously two different aspects of time preferences and that is the reason why this axiom is divided into two parts. *First Order Separability* implies there is no interaction between different periods and *Second Order Separability* is introduced to isolate the impact of time into decision weights. We do not claim for a high level of descriptive power associated to Separability[5] but the main advantage lies in focusing the study on decision weights.

A.3 Impatience

$$a \geq b \geq 0 \Longrightarrow \left(x_{i,j}^{*}a, b\right) \gtrsim \left(x_{i,j}^{*}b, a\right)$$

Impatience says that earlier is always better than later. It is now usual to incorporate impatience in the core properties of discount functions.

Definition 1 *(Discount Functions)*

A discount function $\mu : \mathcal{P} \to]0,1]$ is a function which simultaneously satisfies the three following properties:

(d1) $\mu(0) = 1$ (Normalization)

(d2) $\mu(t_k) > 0$ (Positive discounting)

(d3) μ is (weakly) decreasing with t_k (Impatience)

Property (d1) says that there is no discount for the present period and (d2) implies that the DM has minimal concern for all future periods, no matter how far from the present are. Lastly (d3) exhibits impatience. We are now able to propose the separable representation of preferences on which the main result of this paper will be built. All the proofs are gathered in Appendix.

Theorem 1 (*DU Representation)*

The two following statements are equivalent:

(*) The binary relation $\gtrsim$ satisfies A.1-A.3

(**) There exists a continuous function $U : \Delta \to \mathcal{C}$ such that

$$(x,t) \gtrsim (y,s) \Leftrightarrow U(x,t) \geq U(y,s) \textit{ with } U(x,t) = \sum_{k=0}^{n} \mu(t_k) u(x_k) \quad (1)$$

Where U is *called the present equivalent, μ is a continuous discount function and u is a continuous instantaneous utility function defined to a positive multiple*, *with $u(x^*) = 0$.*

The DU representation (1) has a general form since nothing is said about the nature of discounting for decision weights. The EDU and HDU models can be deduced from the DU model by introducing a specific discounting axiom. The EDU representation needs a stationarity axiom leading to $\mu(i) = \lambda^i$ ($\lambda = e^{-r}$ with $r > 0$) and the HDU representation is built on a decreasing impatience axiom which involves discount function as $\mu(i) = (1+i)^{-r}$ $(r > 0)$ (Harvey (1995) for $b = 1$[6]).

[5] For an exhaustive review of time paradoxes linked with the DU model and especially with separability, see for instance Frederick, Loewenstein and O'Donoghue (2002). The Choquet Utility models as in Chateauneuf and Rébillé (2004) turn separability into sequential separability to capture variation aversion.
[6] Harvey (1995) proposed the following discount function $\mu(i) = (b/(b+i))^r$.

3. The SDU Model

The departure point for the claim of the SDU model is the variety of discount mechanisms pointed out by the experimental literature. For instance Frederick, Loewenstein and O'Donoghue (2002) highlighted that "this literature reveals spectacular variations across (and even within) studies". To top it all, they added that the growing number of studies in the last decades has not produced a diminishing range in elicited discount rates. Such diversity in time discounting is unexplained by the standard DU models since each of them alternatively restricts the attention on a single discount pattern (EDU or HDU). Albrecht and Weber (1995) seminal article proposed for Theorem 1 the following expression for discount weights: for all $i \in \mathcal{P}, \mu(i) = \lambda^{\rho(i)}$ where λ is the standard discount factor ($\lambda = e^{-r}$ with $r > 0$) and ρ a time perception function. Time perception functions transform all real periods into subjective periods and allow distortion of future time intervals. For instance a DM can exhibit a kind of time compression (or concave time perception) so that time perception of the coming year ($i = 12$ if periods are months) is estimated to only six months ($\rho(12) = 6$). Conversely a DM whose time perception of the coming year is estimated to eighteen months will express time extension (or convex time perception). It is important to note that time perception is independent from the nature of time interval (working day, week end or holiday) and only depends on the rank of the period in time ordering. The next definition sharpens the meaning associated with time perception functions through a set of intuitive properties.

Definition 2 *(Time Perception Functions)*

The function $\rho : \mathcal{P} \to \mathcal{P}'$ (where $\mathcal{P}'$ is also a subset of $\mathbb{R}^+$ containing 0) is a time perception function if and only if it satisfies the three following properties:

(p1) $\rho(0) = 0$ (Normalization)

(p2) $\rho(t_k) \geq 0$ (Positive Time)

(p3) $\rho(t_k) < \rho(t_{k+1})$ (Time Order Preserving)

Normalization implies that the present is not submitted to time distortion. Positive Time says that subjective period is still a period and of course the last property ensures that the natural order of time is verified.

Observation 1

For all discount function μ defined as in definition 1, there is a unique time perception function ρ as in definition 2 such that:

$$\forall i \in \mathcal{P}, \ \mu(i) = \lambda^{\rho(i)} \quad (0 < \lambda \leq 1)$$

Observation 1 was first noted by Albrecht and Weber. It is a purely formal representation of time discounting without any axiomatic foundation for time perspective. In other words, subjective time remains an exogenous element in the DU model. We argue that the latter is the main barrier preventing time perception from getting more audience in intertemporal choice. Zauberman and *ali*. (2009) emphasize that the traditional "research […] attribute hyperbolic discounting to changes in the perception or valuation of outcomes at different points of time". The authors claim that hyperbolic discounting evidence has likely to be found from changes in the valuation of prospective time. Therefore time perspective is a key factor to explain how any discount mechanism departs from exponential discounting. In the following, we endogenize subjective time showing how time perspective may be inserted into a behavioral axiom. In the spirit of Albrecht and Weber (1995), we introduce the SDU model whose functional representation satisfies (1) and discounting weights can be written as standard discounting ($0 < \lambda \leq 1$) of the time perception function ρ as previously defined:

$$U(x,t) = \sum_{k=0}^{n} \lambda^{\rho(t_k)} u(x_k) \qquad (2)$$

The next axiom aims at incorporating time perception into the DU model switching over the SDU model which relies on subjective time exponential discounting.

A.4 Subjective Discounting
For all periods $i, j, k \in \mathcal{P}$ ($i < j$ with $\rho(j) + \rho(k) \in \mathcal{P}'$) and for all $a, b \in \mathcal{C}$

$$({x^*}_i a) \sim ({x^*}_j b) \Rightarrow \left({x^*}_{\rho^{-1}(\rho(i)+\rho(k))} a\right) \sim \left({x^*}_{\rho^{-1}(\rho(j)+\rho(k))} b\right)$$

The Subjective Discounting Axiom captures the rich diversity in elicited discount rates since it accommodates any discounting axiom in the DU model. The meaning of A.4 is quite easy to perceive. First, all discount axioms involve objective time in both members of the proposition. Conditions in definition 2 ensure that ρ^{-1} does exist and if ρ converts all real periods in subjective dates then ρ^{-1} produces the reciprocal transformation namely each subjective date is associated to a unique objective period. Furthermore the expression $\rho(i) + \rho(k)$ can be interpreted in a traditional way considering $\rho(k)$ as a common subjective delay. To illustrate, suppose that a 90 immediate reward is the same than obtain 100 in one period (say one year) so $(x_0^* 90) \sim (x_1^* 100)$. Then A.4 implies for a two period decay that $(x_2^* 90) \sim \left(x^*_{\rho^{-1}(\rho(1)+\rho(2))} 100\right)$. Of course, it is not necessary that $\rho^{-1}(\rho(1) + \rho(2)) = 3$. If the case, then $\rho(i) = i$ that is to say time perception is reduced to objective time[7] and A.4 changes for stationarity: $({x^*}_i a) \sim ({x^*}_j b) \Rightarrow ({x^*}_{i+k} a) \sim ({x^*}_{j+k} b)$. The SDU Model reduces to the EDU model with $\mu(i) = \lambda^i$. With the same example, if $\rho(i) = \ln(1 + \alpha i)$ ($\alpha > 0$) where time perception is an extension of the Weber-Fechner law for long term time interval, setting α to 1 gives $\rho^{-1}(\rho(1) + \rho(2)) = 5$. The DM exhibits decreasing impatience and A.4 becomes the central axiom used by Harvey (1995) $({x^*}_i a) \sim ({x^*}_j b) \Rightarrow ({x^*}_{i+k(1+\alpha i)} a) \sim ({x^*}_{j+k(1+\alpha j)} b)$ with $\mu(i) = (1 + \alpha i)^{-r}$.

Theorem 2 *(SDU Representation)*

The two following statements are equivalent:

(*) The binary relation $\succsim$ satisfies A.1-A.4
(**) The representing function of $\succsim$ is a SDU function (2)

The main Theorem completely determines the nature of discounting through the definition of time perception. Therefore it seems natural to link the nature of the discount weight with a specific time perception property. In a nutshell, we will show that the impatience class may be linked with the concavity (or convexity) property of the time perception function. If we assume that SDU decision weights are twice differentiable, then it is possible to use the Prelec's criterion[8] $\gamma(i)$ in order to associate to each discount weight μ a single degree of impatience. A well known result is that for $i \in \mathcal{P} - \{0\}$, decreasing (resp. constant, increasing) impatience holds if and only if $\gamma(i) > 0$ (resp. $= 0, < 0$).

[7] Actually time perception needs to be linear, that is $\rho(t) = m \times t$ with $m > 0$.

[8] Prelec (2004) proposed an "intertemporal Arrow-Pratt" measure $\gamma(i) = -\frac{\mu''(i)}{\mu'(i)} + \frac{\mu'(i)}{\mu(i)}$ for the degree of impatience.

Theorem 3
For a SDU representation of preferences (2), decreasing (resp. constant, increasing) impatience holds if and only if time perception is a concave (resp. linear, convex) function.

We finally present a specific case to illustrate Theorem 2 and Theorem 3. Assume that time perception follows a Steven's power law[9] with $(i) = i^{\beta}$ $(\beta > 0)$. The Subjective Discounting axiom becomes:

$$(x, s) \sim (y, t) \Rightarrow \left(x, \left(s^{\beta} + k^{\beta}\right)^{1/\beta}\right) \sim \left(y, \left(t^{\beta} + k^{\beta}\right)^{1/\beta}\right)$$

The main interest of this case is to consider different impatience degrees depending on the single value of parameter "β" and especially, Steven's time perception allows increasing impatience. This result directly arises from the observation that concavity (convexity) only depends on the value of β. If $\beta = 1$ then the Subjective Discounting Axiom reduces to stationarity and the DM has exponential discounting. To illustrate the case where $\beta < 1$, take for instance the value $\beta = 0{,}5$. Then A.4 becomes as follow $(x, s) \sim (y, t) \Rightarrow (x, i + k + 2(ik)^{0,5}) \sim (y, j + k + 2(jk)^{0,5})$. Thus the axiom departs from stationarity by time premium $\tau(i, k) = 2(ik)^{0,5}$. The function τ increases with time (i and k) so the more rewards are delayed in distant future and the more the DM will be likely to wait for obtaining the greater reward. The opposite interpretation states for $\beta > 1$.

Corollary 1
Assume the SDU model holds with a Steven's time perspective. Then the representative intertemporal utility function has the following form:

$$U(x, t) = \sum_{k=0}^{n} \lambda^{t_k{}^{\beta}} u(x_k) \quad (\beta > 0) \quad (3)$$

Decreasing (resp. constant, increasing) impatience holds if and only if $0 < \beta < 1$ (resp. $\beta = 1, \beta > 1$).

4. Conclusion

The introduction of time perception into standard intertemporal decision methodology provides a sound foundation for explaining the diversity of discount mechanisms. The Subjective Discounting Axiom appears as a simple way to generalize all DU models. Making a step further, time perception might be a key element to resolve the tricky issue opened by Drouhin (2009) explaining why hyperbolic DM could be time consistent by allowing, for instance, concave time perspectives.

Appendix

Proof of Observation 1
For all $i \in \mathcal{P}$ set $f(i) = \lambda^{\rho(i)}$ where ρ is a time perception function. Then (p1) implies (d1) since $f(0) = 1$, $\lambda^{\rho(i)} > 0$ implies (d2) and (p3) leads to (d3) for f decreases with i $(0 < \lambda \leq 1)$. Moreover, (d1), (d2) and (d3) simultaneously ensure that the image set for f is $]0,1]$. Then the uniqueness of time perception is trivial. ■

[9] Stanley Smith Stevens (1906-1973)

Proof of Theorem 1

$((**) \implies (*))(1)$ implies A.1-A.3 is immediate.

$((*) \implies (**))$ For all t in T, by Gorman (1968), A.1 and A.2.a imply the following form for intertemporal utility functions $U_t(x,t) = \sum_{k=0}^{n} v_t(x_k, t_k)$. Each v_t $(t \in T)$ is a continuous function unique up to a linear transformation, v increases with x_k and for all k in S $v_t(x^*, t_k) = v_s(x^*, s_k) = K$ $(K \in \mathcal{C})$. By A.1 $(x_i^* a, s) \sim (x_i^* a, t)$ so $\forall s,t \in T$, $\forall a \in \mathcal{C}$ and $\forall i \in \mathcal{P}, v_s(a,i) = v_t(a,i) \equiv v(a,i)$ and $U(x,t) = \sum_{k=0}^{n} v(x_k, t_k)$. By impatience v decreases with t_k. Moreover, as v is unique up to linear transformation $v'(x_k, t_k) = \alpha v(x_k, t_k) + \beta$ $(\alpha > 0, \beta \geq 0)$ still represents the DM preferences so we are free to choose v' such that $v'(x^*, t_k) = 0$. In this way, if $(x_i^* a) \sim (x_j^* b)$ and $(x_k^* a) \sim (x_l^* b)$ then $v'(a,i) = v'(b,j)$ and $v'(a,k) = v'(b,l)$ and by A.2.b $v'(c,i) = 2v'(a,i)$ implies $v'(c,k) = 2v'(a,k)$ hence $\frac{v'(c,i)}{v'(a,i)} = \frac{v'(c,k)}{v'(a,k)}$. So setting $a = k = 0$ gives $v'(c,i) = \frac{v'(0,i)}{v'(0,0)} \times v'(c,0)$ (if $v'(0,0) \neq 0$ or else take another value for k) which induces $v'(c,i) = m(i) \times u(c)$ for $m(i) = \frac{v'(0,i)}{v'(0,0)}$ and $u(c) = v'(c,0)$ and the continuity of v' implies continuity for m and u. The intertemporal utility function is $U(x,t) = \sum_{k=0}^{n} m(t_k) \times u(x_k)$ and can be interpreted as a present equivalent $(x_0^* U)$ where $u(x^*) = 0$ (since $\forall i,j \in \mathcal{P}, m(i) \times u(x^*) = m(j) \times u(x^*)$). The instantaneous utility function u is increasing with x_k and unique up to a positive multiple. By monotonicity $m(t_k) > 0$ and $m(0) = \frac{v(0,0)}{v(0,0)} = 1$. Lastly Impatience (A.3) implies that m is decreasing with k. Therefore noting that decision weights m are discount weights completes the proof. ■

Proof of Theorem 2

$\big((**) \implies (*)\big)$ The key step of the proof lies on the use of Jensen's functional equation. By Aczél (1966), the most general solution of the Jensen's functional equation $f\left(\frac{x+y}{2}\right) = \frac{f(x)+f(y)}{2}$ for a continuous function f is the linear form $f(x) = mx + p$ (for all x, y such that f is well defined) (Theorem 1 page 43). We extend this well known result to logarithmic compound functions. The following functional equation $\ln g(X) = \frac{1}{2}\ln g(X-k) + \frac{1}{2}\ln g(X+k)$ (4) (for $g(.) > 0$ and for all k such that g is well defined) is a Jensen's functional equation. Indeed set $f(X) = \ln g(X)$ then (4) becomes $f(X) = \frac{1}{2} f(X-k) + \frac{1}{2} f(X+k)$ and replacing X for $\frac{x+y}{2}$ and k for $\frac{y-x}{2}$ will do the job.

According to Theorem 1, axioms A.1 to A.3 induce the functional representation (1). Therefore if $(x^*{}_i a) \sim (x^*{}_j b)$ then Subjective Discounting implies for all $,j,k \in \mathcal{P}$ $\mu(j) \times \mu\left(\rho^{-1}(\rho(i) + \rho(k))\right) = \mu(i) \times \mu\left(\rho^{-1}(\rho(j) + \rho(k))\right)$. Hence choosing $\rho(k) = \rho(j) - \rho(i)$, we obtain $\ln(\mu(j)) = \frac{1}{2}\ln\left(\mu\left(\rho^{-1}(\rho(j) - \rho(k))\right)\right) + \frac{1}{2}\ln\left(\mu\left(\rho^{-1}(\rho(j) + \rho(k))\right)\right)$ that is to say, with $\mu(j) = g(\rho(j))$, $\ln\left(g(\rho(j))\right) = \frac{1}{2}\ln\left(g(\rho(j) - \rho(k))\right) + \frac{1}{2}\ln\left(g(\rho(j) + \rho(k))\right)$ (where $g(j) = \mu[\rho^{-1}(j)]$ so g is well defined). Since $f(j) = \ln\left(g(\rho(j))\right)$ is continuous then the previous functional equation is a Jensen's equation whose broadest solution is $\ln\left(g(\rho(j))\right) = \ln(\mu(j)) = a\rho(j) + b$. As $\mu(0) = 1$ and $\rho(0) = 0$ so $b = 0$. Hence $\ln(\mu(j)) = a\rho(j)$ with $a < 0$ since μ is a discount function. Finally we obtain $\mu(j) = \lambda^{\rho(j)}$ $(\lambda = e^{-r}, r = -a)$.

$\big((*) \implies (**)\big)$ Conversely, SDU verifies (1) so according to Theorem 1, A.1 to A.3 are implied. In addition, if for all $j,k \in \mathcal{P}$ $\mu(j) = \lambda^{\rho(j)}$ hence $\mu\left(\rho^{-1}(\rho(j) + \rho(k))\right) = \lambda^{\rho(j)+\rho(k)}$ then under (1) we have $(x_i^* a) \sim (x_j^* b) \Longrightarrow \left(x^*_{\rho^{-1}(\rho(i)+\rho(k))} a\right) \sim \left(x^*_{\rho^{-1}(\rho(j)+\rho(k))} b\right)$ ■

Proof of theorem 3

First note that for a SDU model, the Prelec's criterion is $(i) = -\frac{\rho''(i)}{\rho'(i)}$ $(i \neq 0)$. Then because time perception is an increasing function of time, $\forall i \in \mathcal{P} - \{0\}$ $\rho'(i) > 0$ so $\gamma(i)$ has the same sign as $-\rho''(i)$. Hence $\rho''(i) < 0$ $(\rho''(i) = 0\,, \rho''(i) > 0\,)$ namely time perception is concave (linear, convex) and $\gamma(i) > 0$ $(\gamma(i) = 0\,, \gamma(i) < 0\,)$ thus impatience is decreasing (constant, increasing). ■

Proof of Corollary 1

Taking $\rho(i) = i^\beta$, by Theorem 2 the representation (3) follows. The Prelec's criterion for $\mu(i) = \lambda^{i^\beta}$ is $\gamma(i) = \frac{1-\beta}{i}$ $(i \neq 0)$ then according to Theorem 3, the proof is immediate. ■

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