Economics Bulletin

Volume 32, Issue 3

# Identities for Homogeneous Utility Functions

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## Abstract

Using a homogeneous and continuous utility function to represent a household's preferences, we show explicit algebraic ways to go from the indirect utility function to the expenditure function and from the Marshallian demand to the Hicksian demand and vice versa, without the need of any other function. This greatly simplifies the integrability problem, avoiding the use of differential equations. In order to get this result, we prove explicit identities between most of the different objects that arise from the utility maximization and the expenditure minimization problems.

We thank Glen Weyl, Federico Echenique and Jan Zapal for their valuable comments, and Gustavo Paez for his research assistance.
**Citation:** Miguel A. Espinosa and Juan D. Prada, (2012) "Identities for Homogeneous Utility Functions", *Economics Bulletin*, Vol. 32 No. 3 pp. 2026-2034.
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**Submitted:** April 28, 2012. **Published:** July 20, 2012.

# 1 Introduction

Although several books of intermediate and advanced microeconomic theory provide some identities between the objects derived from the household's problems, they do not provide explicit ways to go from one object to another one. This paper proposes identities which allow to shift between the different objects associated with household optimization when the utility function is homogeneous, without the need for a third function. For instance, Mas-Colell, Whinston and Green (1995) show that the indirect utility function is necessary in order to obtain the Marshallian demand once one has the Hicksian demand (see their figure 3.G.3). As an alternative example, consider the identity that requires both the Marshallian demand and the expenditure function to obtain the Hicksian demand. This paper tackles the necessity of a third function (i.e., indirect utility or expenditure function) for those and other identities.

Homogeneous utility functions are very easy to deal with theoretically and in applications. The simplest example is given by the CES utility class, which includes Cobb-Douglas and Leontief utility functions as particular cases. These preferences are extensively used in applications, and can be utilized to understand welfare implications of policy changes. Empirically, it is often the case that the Marshallian demands can be recovered using data on prices and quantities. However, to analyze the impact of different policies on welfare, Hicksian demands and/or the utility function are required. The identities proven in this paper allow to recover objects of interest, under the assumption of homogeneity, starting from observable objects, with simple algebraic procedures.

# 2 Theoretical Framework

We let the consumption set to be $\mathbb{R}^n_+$. The preferences of the consumer can be described by a utility function $u : \mathbb{R}^n_+ :\longrightarrow \mathbb{R}$. We will assume one or more of the following conditions:

C.1 $u(\cdot)$ is homogeneous of degree $\gamma$, i.e., for all $t > 0$ and all $\mathbf{x} \in \mathbb{R}^n_+$, $u(t\mathbf{x}) = t^\gamma u(\mathbf{x})$ where $\gamma \geq 0$ is the degree of homogeneity.

C.2 $u(\cdot)$ is strictly increasing in $\mathbf{x}$, i.e., if $\mathbf{x}_1 \gg \mathbf{x}_2$ then $u(\mathbf{x}_1) > u(\mathbf{x}_2)$.

C.3 $u(\cdot)$ is a continuous and quasiconcave function over $\mathbb{R}^n_+$.

Homogeneity and continuity imply that $u(0) = 0$, and C.3 implies that $u(\mathbf{x}) > 0$ for $\mathbf{x} \gg \mathbf{0}$. It is important to note that C.1 and C.2 imply that $u(\cdot)$ is unbounded above. Finally, as shown by Prada (2011), if we assume C.1 with $0 < \gamma \leq 1$, quasiconcavity and that $u(\cdot)$ is increasing, then $u(\cdot)$ is concave.

The expenditure function, expressing the minimum expenditure at which an agent can achieve a fixed level of utility $u \in \mathbb{R}_+$, taking the goods' price vector $\mathbf{p} \in \mathbb{R}^n_{++}$ as given, is denoted by $e(u, \mathbf{p})$. The set of consumption bundles that minimize expenditure, known as the Hicksian demand correspondence, is denoted by $\mathbf{x}^h(u, \mathbf{p})$. It follows that $e(u, \mathbf{p}) = \mathbf{p} \cdot \mathbf{x}^h(u, \mathbf{p})$.

The indirect utility function expresses the maximum utility that the consumer can achieve as a function of wealth and goods' price vector (i.e $m \in \mathbb{R}_+$ and $\mathbf{p} \in \mathbb{R}^n_{++}$). It is denoted as $v(m, \mathbf{p})$. The Marshallian demand correspondence is denoted by $\mathbf{x}(m, \mathbf{p})$.

The following Proposition summarizes some well known relationships among the objects previously defined (see, for example, Mas-Colell, Whinston and Green (1995)).

**Proposition 1** *Let $u(\cdot)$ be a continuous function satisfying C.2. For positive prices, wealth and utility $u > u(\mathbf{0})$ we have*

1. *Walras' law: $\mathbf{p} \cdot x(m, \mathbf{p}) = m$. This holds for any $\mathbf{p}$.*
2. *Demand identities: $x(m, \mathbf{p}) = x^h(v(m, \mathbf{p}), \mathbf{p})$, $x^h(u, \mathbf{p}) = x(e(u, \mathbf{p}), \mathbf{p})$.*
3. *Optimal value function identities: $e(v(m, \mathbf{p}), \mathbf{p}) = m$, $v(e(u, \mathbf{p}), \mathbf{p}) = u$.*
4. *Other identities: $u\left(x^h(u, \mathbf{p})\right) = u$, $\mathbf{p} \cdot x(m, \mathbf{p}) = e(v(m, \mathbf{p}), \mathbf{p})$.*

# 3 Identities Between Representations of a Homogeneous Utility Function

The next result is similar to the one presented in Jehle and Reny (2000), Theorem 2.3. The main difference between these two results is that we use continuity of the utility function (i.e. Assumption C.3.), while Jehle and Reny (2000) use differentiability of the utility function. Since differentiability implies continuity, but continuity does not imply differentiability, our result is more general.

**Proposition 2** *If the utility function satisfies C.2 and C.3, then*

$$u(\mathbf{x}) = \min_{\mathbf{p} \in \mathbb{R}^n_{++}} \{v(m, \mathbf{p}) : \mathbf{p} \cdot \mathbf{x} \leq m\}$$

**Proof.** Fix $\mathbf{x} \in \mathbb{R}^n_{++}$. Define $\tilde{u}(\mathbf{x}) = \min_{\mathbf{p} \in \mathbb{R}^n_{+}} \{v(m, \mathbf{p}) : \mathbf{p} \cdot \mathbf{x} \leq m\}$ and take a minimizer $\mathbf{p}$. Then, we have $\tilde{u}(\mathbf{x}) = v(m, \mathbf{p}) \geq u(\mathbf{x})$ because $\mathbf{p} \cdot \mathbf{x} \leq m$ and because of the definition of $v(m, \mathbf{p})$.

On the other hand, since $u(\mathbf{x})$ is quasiconcave, the upper contour level set is convex. Thus, by the separating hyperplane theorem, there exists a $\mathbf{q} \neq \mathbf{0}$ and a $r \in \mathbb{R}$ such that $\mathbf{q} \cdot \mathbf{x} \leq r \leq \mathbf{q} \cdot \mathbf{y}$ for all $\mathbf{y} \in \mathbb{R}^n_+$ such that $u(\mathbf{y}) \geq u(\mathbf{x})$.

We have $\mathbf{q} \in \mathbb{R}^n_+$. If not, there is some i such that $q_i < 0$. Then, making $\mathbf{y} = \mathbf{x} + \boldsymbol{\epsilon} + \alpha \mathbf{1}_i$ where $\boldsymbol{\epsilon} = \left[\begin{array}{ccc} \epsilon & \ldots & \epsilon \end{array}\right]'$ for some $\epsilon > 0$ and $\mathbf{1}_i = \left[\begin{array}{ccccccc} 0 & \ldots & 0 & 1 & 0 & \ldots & 0 \end{array}\right]'$ we have that for $\alpha > 0$ big enough $\mathbf{q} \cdot \mathbf{y} < r$ and $u(\mathbf{y}) > u(\mathbf{x})$. Then, it follows that $r \geq 0$ since $\mathbf{x} \in \mathbb{R}^n_+$. Note that $\mathbf{x}$ is a solution to the utility maximization problem with prices $\mathbf{q}$ and wealth $r$. Then, $\tilde{u}(\mathbf{x}) \leq v(r, \mathbf{q}) = u(\mathbf{x})$. ■

With this duality result and Proposition 1, we can easily prove the following result:

**Corollary 1** *If the utility function $u(\cdot)$ satisfies C.2 and C.3, it is homogeneous of degree $\gamma$ (satisfies C.1) if and only if the expenditure function $e(u, \mathbf{p})$ is homogeneous of degree $\frac{1}{\gamma}$ in $u$ (and so it can be written as $e(u, \mathbf{p}) = u^{\frac{1}{\gamma}} e(1, \mathbf{p})$).*

**Proof.**

Assume C.1. Take $t > 0$. For all $\mathbf{x} \in \mathbb{R}^n_+$, $u(\mathbf{x}) \geq u$ if and only if $u\left(t^{\frac{1}{\gamma}}\mathbf{x}\right) \geq tu$. Then,

$$\begin{aligned} e(u,\mathbf{p}) &= \min_{\mathbf{x}\in\mathbb{R}^n_+}\{\mathbf{p}\cdot\mathbf{x} : u(x) \geq u\} = \min_{\mathbf{x}\in\mathbb{R}^n_+}\left\{\mathbf{p}\cdot\mathbf{x} : u\left(t^{\frac{1}{\gamma}}\mathbf{x}\right) \geq tu\right\} \\ &= \min_{\tilde{\mathbf{x}}\in\mathbb{R}^n_+}\left\{t^{-\frac{1}{\gamma}}\mathbf{p}\cdot\tilde{\mathbf{x}} : u(\tilde{\mathbf{x}}) \geq tu\right\} = t^{-\frac{1}{\gamma}}\min_{\tilde{\mathbf{x}}\in\mathbb{R}^n_+}\{\mathbf{p}\cdot\tilde{\mathbf{x}} : u(\tilde{\mathbf{x}}) \geq tu\} \\ &= t^{-\frac{1}{\gamma}}e(tu,\mathbf{p}). \end{aligned}$$

Now assume that the expenditure function is homogeneous of degree $\frac{1}{\gamma}$ in $u$. Then, $e(u,\mathbf{p}) = u^{\frac{1}{\gamma}}e(1,\mathbf{p}) = u^{\frac{1}{\gamma}}e(\mathbf{p})$ where we define $e(\mathbf{p}) \equiv e(1,\mathbf{p})$ as the normalized expenditure function. Since $m = e(v(m,\mathbf{p}),\mathbf{p}) = v(m,\mathbf{p})^{\frac{1}{\gamma}}e(\mathbf{p})$ we get

$$v(m,\mathbf{p}) = m^\gamma e(\mathbf{p})^{-\gamma}$$

From Proposition 2 we have

$$\begin{aligned} u(t\mathbf{x}) &= \min_{\mathbf{p}\in\mathbb{R}_+}\left\{m^\gamma e(\mathbf{p})^{-\gamma} : \mathbf{p}\cdot t\mathbf{x} \leq m\right\} \\ &= t^\gamma \min_{\mathbf{p}\in\mathbb{R}_+}\left\{\left(\frac{m}{t}\right)^\gamma e(\mathbf{p})^{-\gamma} : \mathbf{p}\cdot\mathbf{x} \leq \frac{m}{t}\right\} \\ &= t^\gamma u(\mathbf{x}) \end{aligned}$$

Then, we get the homogeneity result on $u(\cdot)$ for $\mathbf{x} \in \mathbb{R}^n_{++}$. But by continuity we can extend this result to any $\mathbf{x} \in \mathbb{R}^n_+$. ■

From the proof of Corollary 1 we obtain another result that it is important on its own.

**Corollary 2** *If the utility function $u(\cdot)$ satisfies C.2 and C.3, it is homogeneous of degree $\gamma$ (satisfies C.1) if and only if the indirect utility function $v(m,\mathbf{p})$ is homogeneous of degree $\gamma$ in $m$ (and so it can be written as $v(m,\mathbf{p}) = m^\gamma v(1,\mathbf{p})$).*

**Proof.** It follows from Corollary 1 and the identity $m = e(v(m,\mathbf{p}),\mathbf{p})$. ■

Let $\frac{e(u,\mathbf{p})}{u}$ be the average expenditure, and let $\frac{\partial e(u,\mathbf{p})}{\partial u}$ be the marginal expenditure. If the utility function satisfies C.2, then the expenditure function is homogeneous of degree $\frac{1}{\gamma}$ in $u$ if and only if the ratio of average to marginal expenditure equals $\gamma$. We have an analogous result using the indirect utility function: $\gamma = \frac{\partial v(m,\mathbf{p})}{\partial m}/\frac{v(m,\mathbf{p})}{m}$.

We now show a standard result about the inverse demand correspondence.

**Proposition 3** *Let $u(\cdot)$ be a differentiable utility function that satisfies conditions C.1 to C.3. If the solution to the utility maximization problem is unique and interior, then the inverse demand correspondence is single valued. Also, the inverse demand function can be written as $p_i(m,\mathbf{x}) = \frac{m}{\gamma}\frac{\frac{\partial u(\mathbf{x})}{\partial x_i}}{u(\mathbf{x})}$ for $i = 1,\ldots,n$.*

**Proof.**

We need to prove that the Marshallian demand function is one-to-one. Fix price vectors $\mathbf{p}$ and $\mathbf{p}'$ and assume that $\mathbf{x} = \mathbf{x}(m,\mathbf{p}) = \mathbf{x}(m,\mathbf{p}')$. The first-order conditions for the utility maximization problem give us $\frac{\partial u(\mathbf{x})}{\partial x_i} = \lambda p_i$ and $\frac{\partial u(\mathbf{x})}{\partial x_i} = \lambda' p_i'$ for some constants $\lambda > 0$ and $\lambda' > 0$. Then, we have that for all $i = 1, \ldots, n$, $p_i = \frac{\lambda'}{\lambda} p_i'$. Finally, by monotonicity of the utility function we have $\mathbf{p} \cdot \mathbf{x} = \mathbf{p}' \cdot \mathbf{x} = m$. Then, $\lambda = \lambda'$ and therefore $\mathbf{p} = \mathbf{p}'$. This shows that the Marshallian demand function is invertible.

Now fix $\mathbf{x} \in \mathbb{R}^n_+$. By the envelope theorem we have $\frac{\partial u(\mathbf{x})}{\partial x_i} = \lambda p_i(m, \mathbf{x})$. Multiplying by $x_i$ and summing over $i$ we obtain

$$\sum_{i=1}^{n} x_i \frac{\partial u(\mathbf{x})}{\partial x_i} = \sum_{i=1}^{n} \lambda p_i(m, \mathbf{x}) x_i = \lambda m$$

where we used Walras' law. Then, $p_i = m \frac{\frac{\partial u(\mathbf{x})}{\partial x_i}}{\sum_{i=1}^{n} x_i \frac{\partial u(\mathbf{x})}{\partial x_i}}$. Finally, note that the utility function is homogeneous of degree $\gamma$. Then, $\sum_{i=1}^{n} x_i \frac{\partial u(\mathbf{x})}{\partial x_i} = \gamma u(\mathbf{x})$. ■

A related result can be found in other sources, as Jehle and Reny (2000), Theorem 2.4. The main differences between Jehle and Reny's result and Proposition 3 is that our result works for any income level $m$ while their theorem is for $m = 1$, we use homogeneity of the utility function while they do not, and their argument is based on the envelope theorem while ours is based on the invertibility of the Marshallian demand.

Now we present and prove some identities that reduce the computational burden of shifting from one representation of preferences to another. Let $v(\mathbf{p}) \equiv v(1, \mathbf{p})$ and $e(\mathbf{p}) \equiv e(1, \mathbf{p})$.

Remember that Shephard's lemma and Roy's identity are valid if the solutions to the household's optimization problems are unique. When we use these results we are implicitly assuming uniqueness. Let $\mathbf{p}(m, \mathbf{x})$ denote the inverse demand function.

**Theorem 1** *If the utility function satisfies conditions C.1 to C.3, and the optimal value functions are differentiable in their parameters, then the next identities hold for all $u$, $m \in \mathbb{R}_+$:*

*I.1.* $v(m, \mathbf{p}) = \left[\frac{m}{e(u,\mathbf{p})}\right]^{\gamma} u$. *I.1'.* $v(m, \mathbf{p}) = \left(\frac{m}{e(\mathbf{p})}\right)^{\gamma}$ *and* $v(m, \mathbf{p}) = \left[\frac{m}{\mathbf{p}\cdot\mathbf{x}^h(u,\mathbf{p})}\right]^{\gamma} u$.

*I.2.* $v(m, \mathbf{p}(m, \mathbf{x})) = u(\mathbf{x})$.

*I.3.* $u(\mathbf{x}) = \left[\frac{m}{e(u,\mathbf{p}(m,\mathbf{x}))}\right]^{\gamma} u$. *I.3'.* $u(\mathbf{x}) = \left[\frac{1}{e(\mathbf{p}(1,\mathbf{x}))}\right]^{\gamma}$.

*I.4.* $x_i(m, \mathbf{p}) = -\frac{\partial v(m,\mathbf{p})}{\partial p_i} / \frac{\gamma v(m,\mathbf{p})}{m}$.

*I.5.* $x_i(m, \mathbf{p}) = \frac{m x_i^h(u,\mathbf{p})}{\mathbf{p}\cdot\mathbf{x}^h(u,\mathbf{p})}$. *I.5'.* $x_i(m, \mathbf{p}) = \frac{m \frac{\partial e(u,\mathbf{p})}{\partial p_i}}{e(u,\mathbf{p})} = \frac{m \frac{\partial e(\mathbf{p})}{\partial p_i}}{e(\mathbf{p})}$.

*I.6.* $x_i^h(u, \mathbf{p}) = \frac{u^{\frac{1}{\gamma}} x_i(1,\mathbf{p})}{v(\mathbf{p})^{\frac{1}{\gamma}}} = \frac{u^{\frac{1}{\gamma}} x_i(m,\mathbf{p})}{v(m,\mathbf{p})^{\frac{1}{\gamma}}}$. *I.6'.* $v(m, \mathbf{p}) = \left[\frac{x_i(m,\mathbf{p})}{x_i^h(u,\mathbf{p})}\right]^{\gamma} u = \left[\frac{x_i(m,\mathbf{p})}{x_i^h(1,\mathbf{p})}\right]^{\gamma}$.

*I.7.* $x_i^h(u, \mathbf{p}) = x_i(m, \mathbf{p}) \left(\frac{u}{u(\mathbf{x}(m,\mathbf{p}))}\right)^{\frac{1}{\gamma}} = x_i(1, \mathbf{p}) \left(\frac{u}{u(\mathbf{x}(1,\mathbf{p}))}\right)^{\frac{1}{\gamma}}$.

*I.8.* $e(u, \mathbf{p}) = m \left(\frac{u}{u(\mathbf{x}(m,\mathbf{p}))}\right)^{\frac{1}{\gamma}} = \left(\frac{u}{u(\mathbf{x}(1,\mathbf{p}))}\right)^{\frac{1}{\gamma}}$.

*I.9.* $x_i^h(u, \mathbf{p}) = -\frac{m u^{\frac{1}{\gamma}}}{\gamma} \frac{\frac{\partial v(m,\mathbf{p})}{\partial p_i}}{v(m,\mathbf{p})^{\frac{1+\gamma}{\gamma}}}$.

*I.10.* $x_i(m, \mathbf{p}) = x_i^h\left(\left[\frac{m}{\mathbf{p}\cdot\mathbf{x}^h(u,\mathbf{p})}\right]^{\gamma} u, \mathbf{p}\right) = x_i^h\left(\left[\frac{m}{\mathbf{p}\cdot\mathbf{x}^h(1,\mathbf{p})}\right]^{\gamma}, \mathbf{p}\right)$.

# 4 An algebraic solution to the integrability problem

In this section we show and prove the main result of the paper: simple algebraic identities to obtain Hicskian demands from Marshallian demands and vice versa, without the need of additional functions.

## 4.1 Recovering the Hicksian demands directly from Marshallian demands

From identity I.5 we have $x_i(1,\mathbf{p})\,e(\mathbf{p}) = x_i^h(1,\mathbf{p})$. But the expenditure function can be written as $e(\mathbf{p}) = \mathbf{p}'\mathbf{x}^h(1,\mathbf{p})$. Therefore we get the system of equations $\mathbf{x}^h(1,\mathbf{p}) = \mathbf{x}(1,\mathbf{p})\,\mathbf{p}'\mathbf{x}^h(1,\mathbf{p})$ and the Hicksian demand satisfies the matrix identity

$$[I_n - \mathbf{x}(1,\mathbf{p})\,\mathbf{p}']\,\mathbf{x}^h(1,\mathbf{p}) = \mathbf{0}.$$

Note that by Sylvester's determinant theorem we have $\det[I_n - \mathbf{x}(1,\mathbf{p})\,\mathbf{p}'] = \det[1 - \mathbf{p}'\mathbf{x}(1,\mathbf{p})] = 0$, because $\mathbf{p}'\mathbf{x}(1,\mathbf{p}) = 1$ by Walras' law. Since we must have $\mathbf{x}^h(1,\mathbf{p}) \neq \mathbf{0}$, for homogeneous utility functions the Hicksian demand is one of the nonnegative eigenvectors of $\mathbf{x}(1,\mathbf{p})\,\mathbf{p}'$ associated with the eigenvalue of value one. The system of equations given by this eigenvector identity is of rank $n-1$ and we need another equation to pin down the Hicksian demand.

Because of the normalizations used, we can uniquely pick the eigenvalue that satisfies $\mathbf{p}'\mathbf{x}^h(1,\mathbf{p}) = 1$ and this will give us the Hicksian demand solely from the Marshallian demand, without the need of any other function.

Thus, we can easily solve the integrability problem for continuous, monotone and quasi-concave homogeneous utility functions.

## 4.2 Recovering the Marshallian demands directly from Hicksian demands

The usual way to recover Marshallian demands from Hicksian demands is to find the expenditure function using the definition $e(u,\mathbf{p}) = \mathbf{p}\cdot\mathbf{x}^h(u,\mathbf{p})$, then obtain an indirect utility function and use the identity $\mathbf{x}(m,\mathbf{p}) = \mathbf{x}^h(v(m,\mathbf{p}),\mathbf{p})$. Identities I.5 or I.10 summarize all this procedure in easy algebraic formulas.

We can recover the Marshallian demand for the $i$-th good using identity I.10: $x_i(m,\mathbf{p}) = x_i^h\left(\left[\frac{m}{\mathbf{p}\cdot\mathbf{x}^h(1,\mathbf{p})}\right]^{\gamma},\mathbf{p}\right)$. Here, $x_i^h(\cdot)$ denotes the Hicksian demand for the $i$-th good. This identity allow us to recover the Marshallian demand directly from the vector of Hicksian demands without the use of a third function. Note that identity I.5 also gives us the Marshallian demands.

Thus, we have purely algebraic ways to shift between the different kinds of demand. Note that we require the knowledge of the full vector of demands (i.e. $\mathbf{x}(m,\mathbf{p})$ or $\mathbf{x}^h(u,\mathbf{p})$).

## 4.3 Integrability

The integrability process is simplified as follows:

1. Starting from the Marshallian demand function, recover the Hicksian demand function as shown above.

2. Given the Hicksian demand function, obtain the expenditure function using the identity $e(u, \mathbf{p}) = \mathbf{p} \cdot \mathbf{x}^h(u, \mathbf{p})$.

3. Given the expenditure function $e(u, \mathbf{p})$ obtain, the indirect utility function $v(m, \mathbf{p})$ using I.1 from Theorem 1.

4. Given the indirect utility function $v(m, \mathbf{p})$, recover the utility function $u(\mathbf{x})$ using Proposition 2. Alternatively, given the Marshallian demand we could find $\mathbf{p}(m, \mathbf{x})$ and then use the identity $u(\mathbf{x}) = v(m, \mathbf{p}(m, \mathbf{x}))$.

An important issue is the lack of uniqueness in the solution. If an specific $u(\mathbf{x})$ gives as solution of the utility maximization problem $\mathbf{x}(m, \mathbf{p})$, whatever monotonic and positive transformation to the utility function will give the same set of Marshallian demands $\mathbf{x}(m, \mathbf{p})$. However with the integrability process proposed in this subsection, we will be able to pin down a specific utility function that is homogeneous of degree $\gamma$.

# 5 Conclusions

In this paper, we derived identities that allow to shift between six different ways of representing a homogeneous utility function.

These results are useful to simplify computational procedures when different representations of a utility function are required. For example, we got a simple algebraic formula to shift from the indirect utility function to the expenditure function and vice versa. As far as we know, this useful identity has been ignored in the literature.

Finally, we proved an explicit algebraic way to get Hicksian demands from Marshallian demands and vice versa, without the need of any other function. This allow us to avoid solving differential equations to find the expenditure function, thus simplifying the integrability problem. Note however that these algebraic identities require the knowledge of the full vector of demands (i.e. $\mathbf{x}(m, \mathbf{p})$ or $\mathbf{x}^h(u, \mathbf{p})$). It remains to be analyzed under what conditions can the $i$-th Marshallian demand be recovered using only the $i$-th Hicksian demand and vice versa.

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# A Proof of Theorem 1

From Corollary 1 follows immediately that $e(u,\mathbf{p}) = u^{\frac{1}{\gamma}} e(\mathbf{p})$. Since $m = e(v(m,\mathbf{p}),\mathbf{p})$, we have $m = v(m,\mathbf{p})^{\frac{1}{\gamma}} e(\mathbf{p}) = v(m,\mathbf{p})^{\frac{1}{\gamma}} u^{-\frac{1}{\gamma}} e(u,\mathbf{p})$ and we get I.1 and I.1'.

We obtain I.2 simply by evaluating the inverse demand function in the indirect utility function. This identity follows from Proposition 2.

Identity I.3 can be obtained using I.2 and I.1. Identity I.3' follows in the next way:

$$u(\mathbf{x}) = \left[\frac{m}{e(u,\mathbf{p}(m,\mathbf{x}))}\right]^{\gamma} u = \left[\frac{m}{u^{\frac{1}{\gamma}} e(1,\mathbf{p}(m,\mathbf{x}))}\right]^{\gamma} u = \left[\frac{m}{e(\mathbf{p}(m,\mathbf{x}))}\right]^{\gamma}$$

To prove I.4, note that in the discussion following Corollary 2 we argued that $\gamma = \frac{\frac{\partial v(m,\mathbf{p})}{\partial m}}{\frac{v(m,\mathbf{p})}{m}}$, and by Roy's Identity, Marshallian demands can be expressed as $x_i(m,\mathbf{p}) = -\frac{\partial v(m,\mathbf{p})}{\partial p_i} / \frac{\gamma v(m,\mathbf{p})}{m}$.

In order to prove I.5 note that from I.1,

$$\frac{\partial v(m,\mathbf{p})}{\partial p_i} = m^{\gamma} \frac{\partial}{\partial p_i} e(\mathbf{p})^{-\gamma} = -\gamma m^{\gamma} e(\mathbf{p})^{-\gamma-1} \frac{\partial}{\partial p_i} e(\mathbf{p}) = \frac{-\gamma v(m,\mathbf{p}) x_i^h(1,\mathbf{p})}{e(\mathbf{p})}$$

where $\frac{\partial}{\partial p_i} e(\mathbf{p}) = x_i^h(1,\mathbf{p})$ follows from Shephard's lemma. Then using I.4 we get

$$x_i(m,\mathbf{p}) = -\left(\frac{-\gamma v(m,\mathbf{p}) x_i^h(1,\mathbf{p})}{e(\mathbf{p})}\right) / \frac{\gamma v(m,\mathbf{p})}{m} = \frac{m x_i^h(1,\mathbf{p})}{e(\mathbf{p})}.$$

Note now that since for all price vectors $\mathbf{p}$ we have $\mathbf{p} \cdot \mathbf{x}^h(u,\mathbf{p}) = e(u,\mathbf{p}) = u^{\frac{1}{\gamma}} e(1,\mathbf{p}) = u^{\frac{1}{\gamma}} \mathbf{p} \cdot \mathbf{x}^h(1,\mathbf{p})$ then we must have $\mathbf{x}^h(u,\mathbf{p}) = u^{\frac{1}{\gamma}} \mathbf{x}^h(1,\mathbf{p})$ by continuity of the dot product and because $u$ is scalar [1]. Therefore the Hicksian demand is homogeneous of degree $\frac{1}{\gamma}$ in $u$. Using this homogeneity we get $x_i(m,\mathbf{p}) = \frac{m x_i^h(u,\mathbf{p})}{e(u,\mathbf{p})}$.

Identity I.6 is obtained by a simple algebraic manipulation of I.5' and taking into account that $v(\mathbf{p}) = e(\mathbf{p})^{-\gamma}$ and the homogeneity of the Hicksian demand with respect to $u$. Identity I.6' follows from I.6.

Identity I.7 can be obtained by using I.6 and the definition of the indirect utility function as $v(m,\mathbf{p}) = u(\mathbf{x}(m,\mathbf{p}))$.

Identity I.8 follows from I.7 and Walras' law. We have

$$e(u,\mathbf{p}) = \sum_{i=1}^{n} p_i x_i^h(u,\mathbf{p}) = \sum_{i=1}^{n} p_i x_i(m,\mathbf{p}) \left(\frac{u}{u(\mathbf{x}(m,\mathbf{p}))}\right)^{\frac{1}{\gamma}} = m \left(\frac{u}{u(\mathbf{x}(m,\mathbf{p}))}\right)^{\frac{1}{\gamma}}$$

[1] Take any pair of vectors $\mathbf{x} \in \mathbb{R}^n$ and $\mathbf{y} \in \mathbb{R}^n$. If for all $\mathbf{z} \in \mathbb{R}^n_{++}$, $\mathbf{z} \cdot \mathbf{x} = \mathbf{z} \cdot \mathbf{y}$, then it must be that $\mathbf{x} = \mathbf{y}$. To see this, assume that $\mathbf{x} \neq \mathbf{y}$. There exists a coordinate $i$ such that $x_i \neq y_i$. Choose $\mathbf{z}(i,\varepsilon) = [\varepsilon, \ldots, \varepsilon, 1, \varepsilon, \ldots, \varepsilon]'$, with $\varepsilon > 0$ and the 1 in the $i$-th position. For $\varepsilon$ small enough, $\mathbf{z}(i,\varepsilon) \cdot \mathbf{x} \neq \mathbf{z}(i,\varepsilon) \cdot \mathbf{y}$, by continuity of the dot product.

Identity I.9 can be obtained by using I.1 and Shephard's lemma. First note that from I.1 $e(u, \mathbf{p}) = m \left[ \frac{u}{v(m, \mathbf{p})} \right]^{\frac{1}{\gamma}}$. Then, taking derivative with respect to $p_i$ we find

$$x_i^h(u, \mathbf{p}) = \frac{\partial e(u, \mathbf{p})}{\partial p_i} = -\frac{1}{\gamma} m u^{\frac{1}{\gamma}} v(m, \mathbf{p})^{-\frac{1+\gamma}{\gamma}} \frac{\partial v(m, \mathbf{p})}{\partial p_i}$$

which is the same as $x_i^h(u, \mathbf{p}) = -\frac{mu^{\frac{1}{\gamma}}}{\gamma} \frac{\frac{\partial v(m,\mathbf{p})}{\partial p_i}}{v(m,\mathbf{p})^{\frac{1+\gamma}{\gamma}}}$.

Finally, to get I.10, remember that $x_i(m, \mathbf{p}) = x_i^h(v(m, \mathbf{p}), \mathbf{p})$. Then, the identity follows by substituting I.1.