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A simple decentralized matching mechanism in markets with couples

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Abstract

We analyze a simple decentralized matching mechanism in market with couples called One Application Mechanism. Under this mechanism any stable matching of the market can be attained in Subgame Perfect equilibrium (SPE). In contrast with previous results, we find that the mechanism may attain unstable matchings in SPE. We show that only one special kind of instability is admissible in equilibrium and we argue that this exclusively comes from coordination failures between members of couples. Our main result shows that the One Application Mechanism implements in SPE the set of pairwise stable matchings in markets with couples.

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1 Introduction.

Two main features characterize real life matching markets. First of all, most of them are decentralized, i.e. the final allocations in the market are the outcome of actions of individual decision-makers. Secondly, around the world the number of couples searching jointly for a job has been increasing in recent years (Klaus, Klijn and Massó, 2007).

This paper deals with the analysis of decentralized matching mechanisms in job matching markets with couples. In this setting, we analyze a simple mechanism called **One Application Mechanism ($\mathcal{OA}$)**, previously introduced in the context of college admissions problems[1] (Alcalde, Pérez-Castrillo and Romero-Medina, 1998). Recent literature[2] suggests that stable matchings may be expected equilibrium outcomes when agents face strategically decentralized matching mechanisms in markets with couples.

Under the assumption that couples' preferences are responsive, we fully characterize the set of Subgame Perfect Equilibrium (SPE) outcomes of the game induced by the $\mathcal{OA}$ in matching markets with couples. We show that any stable matching of the market can be attained as a SPE outcome of the game, but in contrast with previous findings we also show that unstable matchings may be supported in SPE. We prove that only one special kind of instability is reasonable in equilibrium, and furthermore we show that this instability of equilibrium outcomes comes from coordination failures between members of couples.

Our main result shows that the $\mathcal{OA}$ implements in SPE the set of pairwise stable matchings in markets with couples. Finally, our characterization provides evidence that decentralized mechanisms work very well and better than centralized mechanisms for some instances of the problem.

The rest of the paper is organized as follows. In Section 2, we describe the basic model; in Section 3, we analyze the matching mechanism and the problem of implementation; in Section 4, we briefly compare decentralized and centralized matching procedures in markets with couples; in Section 5, we present some conclusions. All proofs are in the appendix, in Section 6.

2 The model.

Let $H = \{h_1, ..., h_n\}$ be a set of n hospitals and let $S = \{s_1, ..., s_m\}$ be a set of m medical students. Assume that each hospital has only one position and $|H| = |S| \geq 2k$ for any $k \geq 1$. A couple is an unordered pair of students, we denote a typical couple by a pair $c = (s_k, s_l)$ such that $s_k, s_l \in S$ and $s_k \neq s_l$. Assume that there are no single students, then each student belongs to at most one couple. Let $C = \{c_1, ..., c_k\}$ denote the set of couples in the market and let $\emptyset$ and u be, respectively, the hospital's option of having its position unfilled and the student's prospect of being unemployed. Let $\mathcal{H} = \{H \cup \{u\} \times H \cup \{u\}\} \setminus \{(h, h) : h \in H\}$ denote the set of all possible ordered pairs of hospitals and the prospect of being unemployed. A typical element of $\mathcal{H}$ is denoted by (h_p, h_q).

Each hospital $h \in H$ has a strict, transitive and complete preference relation P_h on the set $S \cup \{\emptyset\}$. Let R_h denote the weak preference relation induced by P_h, so for any pair of students $s, s' \in S$ the relation sR_hs' implies either sP_hs' or $s = s'$. Let $P^H = \{P_h\}_{h\in H}$ denote the preference profile of hospitals. Each couple $c \in C$ has a strict, transitive and complete preference relation P_c on the set $\mathcal{H}$. In a similar way, we denote by R_c the weak preference relation induced by P_c. Let $P^C = \{P_c\}_{c\in C}$ denote the preference profile of couples. In the following, we fix the sets of hospitals and medical students H and S, then a market with couples is completely described by a tuple$\left(P^H, P^C\right)$.

In order to simplify, we focus on the simplest one-to-one matching problem. Formally, a matching between hospitals and medical students is defined as follows.

[1] See Roth and Sotomayor (1990) for a good survey on two-sided matching literature.

[2] For instance, Alcalde, Pérez-Castrillo and Romero-Medina (1998); Alcalde and Romero-Medina (2000); Triossi (2009) and Haeringer and Wooders (2011).

Definition 1 *A* ***matching*** μ *is a mapping from* $H \cup S$ *into* $H \cup S$ *such that:*

1. *for all* $s \in S$*, if* $\mu(s) \notin H$ *implies that* $\mu(s) = u$*;*

2. *for all* $h \in H$*, if* $\mu(h) \notin S$ *implies that* $\mu(h) = \emptyset$*; and*

3. $\mu(s) = h$ *if and only if* $\mu(h) = s$*.*

Let $M(P^H, P^C)$ denote the set of all possible matchings in the market (P^H, P^C). Now we introduce the usual notion of stability in markets with couples. First of all, we define the concept of **individually rational** matchings.

Definition 2 *A matching* μ *is* ***individually rational*** *if,*

1. *for all* $c = (s_k, s_l) \in C$*,* $(\mu(s_k), \mu(s_l)) R_c (\mu(s_k), u)$*,* $(\mu(s_k), \mu(s_l)) R_c (u, \mu(s_l))$ *and* $(\mu(s_k), \mu(s_l)) R_c (u, u)$*; and*

2. *for all* $h \in H$*,* $\mu(h) R_h \emptyset$*.*

Secondly, we define the concept of **blocking coalitions.**

Definition 3 *The coalition* $(c = (s_k, s_l), (h_p, h_q))$ *blocks the matching* μ *if,*

1. $(h_p, h_q) P_c (\mu(s_k), \mu(s_l))$ *and*

2. $h_p \in H$ *implies* $s_k R_{h_p} \mu(h_p)$ *and* $h_q \in H$ *implies* $s_l R_{h_q} \mu(h_q)$*.*

A matching μ is **stable** if it is individually rational and not blocked by any coalition. Let $S(P^H, P^C) \subset M(P^H, P^C)$ denote the set of stable matchings of the market (P^H, P^C).

It is well known that the set of stable matchings may be empty in markets with couples (Roth and Sotomayor, 1990). To assure the existence of at least one stable matching, we assume that couples' preferences are **responsive**[3] (Klaus and Klijn, 2005; Klaus, Klijn and Nakamura, 2009). Formally,

Definition 4 *A couple* $c = (s_k, s_l)$ *has* ***responsive preferences*** *if there exists individual preferences* $\succ_{s_k}$ *and* $\succ_{s_l}$ *such that for all* $h_p, h_q, h_r \in H \cup \{u\}$*,*

1. $h_p \succ_{s_k} h_r$ *implies* $(h_p, h_q) P_c (h_r, h_q)$ *and*

2. $h_p \succ_{s_l} h_r$ *implies* $(h_q, h_p) P_c (h_q, h_r)$*.*

If the preference orders $\succ_{s_k}$ *and* $\succ_{s_l}$ *exist, then they are unique.*

Since the paper deals with a problem of implementation, we need a description of this tool. The literature on implementation theory is well known[4], however in the next lines we briefly describe the notion of extensive form matching mechanisms and the concept of implementation in Subgame Perfect Equilibrium (SPE). An **extensive form matching mechanism** (Triossi, 2009) is an array $G = \langle S \cup H, K, A, g \rangle$. $S \cup H$ is the set of players, K is the set of histories and A is the strategy space. Let Z be the set of terminal histories. Given the initial history k^0, any strategy profile $a \in A$ defines a unique terminal history $z_a \in Z$. Let $g : Z \to M(P^H, P^C)$ be the outcome function, this function specifies an outcome matching for each terminal history.

[3] There are weaker domains of preferences that assure the existence of stable matchings. For instance, the weakly responsive preferences (Klaus and Klijn, 2005; Klaus, Klijn and Nakamura, 2009; Nakamura, 2005). However, all relevant results in markets with couples, i.e. the existence of stable matchings, the lost of the lattice structure and the non-existence of the (hospital) student-optimal stable matching, hold under responsive preferences. Then we restrict our analysis on this domain of preferences.

[4] See Maskin, E. and Sjostrom, T. (2002).

A SPE is a strategy profile a^* that induces a Nash equilibrium in every subgame. Let $z_{a^*} \in Z$ be the terminal history induced by a SPE a^*. The matching $g(z_{a^*})$ is called SPE outcome of the extensive form game induced by G, let $SPE(G)$ denote the set of SPE outcomes of G. Let $\mathcal{S}$ be the set of matching markets and let $\Phi : \mathcal{S} \twoheadrightarrow \Phi$ be a matching correspondence. An extensive form matching mechanism G implements Φ in SPE if for every market $(P^H, P^C) \in \mathcal{S}$, $SPE(G) = \Phi(P^H, P^C)$.

3 The One Application Mechanism.

In this section, we analyze a very simple decentralized matching mechanism previously introduced in the context of college admissions problems (Alcalde, Pérez-Castrillo and Romero-Medina, 1998; Alcalde and Romero-Medina, 2000). Even when this mechanisms is very simple, it mimics many real life matching procedures and makes possible to analyze the strategic behavior of agents who face decentralized matching procedures.

This mechanism is called **One Application Mechanism** ($\mathcal{OA}$) and formally runs as follows:

1. Application: Each student $s \in S$ sends a message, $m(s) \in H \cup \{u\}$, where $m(s) = u$ implies that the student s prefers to remain unemployed and $m(s) \in H$ implies that s applies to some hospital $h \in H$. Let $M(h)$ denote the set of students who apply to the hospital h;

2. Hiring: Each hospital $h \in H$ considers its proposers $M(h)$ and the option of having its position unfilled $\emptyset$. Hospitals choose an element in $M(h) \cup \{\emptyset\}$. Let $J_h(M(h) \cup \{\emptyset\})$ denote the election of the hospital h from the set $M(h) \cup \{\emptyset\}$.

Since students send at most one application and each hospital chooses at most one candidate, the outcome of the $\mathcal{OA}$ is a well defined matching. The $\mathcal{OA}$ induces a game in extensive form denoted by Γ^{OA}, where $S \cup H$ is the set of players. At each step of the game agents play simultaneously. In previous literature, it is found that the $\mathcal{OA}$ implements in SPE the core correspondence of college admissions problems (Alcalde, Pérez-Castrillo and Romero-Medina, 1998; Alcalde and Romero-Medina, 2000).

We analyze the $\mathcal{OA}$ in the context of markets with couples. Note that this extension is not direct. First of all, a market with couples is fundamentally different to the standard matching problem. Even in the simplest setting (the one-to-one problem) the existence of stable matchings is not guaranteed (Roth and Sotomayor, 1990; Gale and Shapley, 1962). Then, we need to consider constrains on couples' preferences to assure the existence of stable matchings.

Second, when agents face strategically matching mechanisms, they usually evaluate outcomes and possible deviations through individual preferences. The presence of couples introduces an additional problem. In markets with couples, we observe couples' preferences but we do not usually observe individual preferences, since the welfare of an individual is not independent of the welfare of her partner. Hence, we have to analyze a problem of individual strategic behavior in a setting where agents evaluate outcomes (and possible deviations) through couples' preferences. In this sense, we can consider the members of a couple as players whose actions are strategic complements.

We incorporate these characteristics of markets with couples in our analysis. First we introduce some additional notation. Given any market (P^H, P^C), we define for each hospital $h \in H$ the h's choice function as follows.

Definition 5 *For any $S' \subset S$, the h's choice function $C_h : S' \cup \{\emptyset\} \to S' \cup \{\emptyset\}$ satisfies the following:*

1. $C_h(S' \cup \{\emptyset\}) \in S' \cup \{\emptyset\}$ *and*
2. $C_h(S' \cup \{\emptyset\}) R_h x$ *for all* $x \in S' \cup \{\emptyset\}$.

We are interested in characterizing the set of SPE outcomes (in pure strategies) of the game Γ^{OA}. First of all, we analyze the strategic behavior of hospitals. In this game, it is clear that each hospital has a dominant strategy that coincides with the decision rule $J_h(\cdot) = C_h(\cdot)$. Clearly this rule is optimal and independent of the strategies of all other agents. It is obvious that, at the final stage of the $\mathcal{OA}$, each hospital cannot do anything better than choosing the best applicant among the ones who have applied to. Clearly, at any SPE of the game Γ^{OA}, hospitals follow their dominant strategy $C_h(\cdot)$.

Given the profile of optimal decision rules for hospitals $J^* = \{C_h(\cdot)\}_{h\in H}$, the $\mathcal{OA}$ induces a n-players game in strategic form $G^{OA} = \left(S, \{H \cup \{s\}\}_{s\in S}, P^C\right)$ played by medical students. Note that the extensive form game Γ^{OA} has a SPE in pure strategies if and only if the corresponding strategic form game G^{OA} has a Nash equilibrium in pure strategies. Observe that under the $\mathcal{OA}$, any equilibrium in pure strategies of the game Γ^{OA} yields a well defined matching, otherwise students play mixed strategies and choose a probability distribution over the set of actions $H\cup\{s\}$ and no well defined matching is attained in equilibrium.

Before establishing our main results, we analyze a simple example to show the difficulties of the problem that come from the strategic behavior of medical students. Consider the following market $\left(P^H, P^C\right)$ with four hospitals and four students. There are two couples in the market: $c_1 = (s_1, s_2)$ and $c_2 = (s_3, s_4)$. Hospitals' and couples' preferences are described in Table 1. Assume that couples' preferences are completed to be strictly unemployment averse, i.e. for each $c \in C$ and all $h_p, h_q \neq u$, it is satisfied $(h_p, h_q)\, P_c\, (h_p, u)\, P_c\, (u, u)$ and $(h_p, h_q)\, P_c\, (u, h_q)\, P_c\, (u, u)$.

Table 1

Hospitals' Preferences				Couples' Preferences	
P^H				P^C	
h_1	h_2	h_3	h_4	$c_1 = (s_1, s_2)$	$c_2 = (s_3, s_4)$
s_4	s_4	s_2	s_2	(h_1, h_2)	(h_4, h_2)
s_2	s_3	s_3	s_4	(h_4, h_1)	(h_4, h_3)
s_1	s_2	s_1	s_1	(h_4, h_3)	(h_4, h_1)
s_3	s_1	s_4	s_3	(h_4, h_2)	(h_3, h_1)
$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	(h_1, h_4)	(h_3, h_2)
				(h_1, h_3)	(h_3, h_4)
				(h_3, h_4)	(h_2, h_4)
				(h_3, h_1)	(h_2, h_1)
				(h_3, h_2)	(h_2, h_3)
				(h_2, h_3)	(h_1, h_2)
				(h_2, h_4)	(h_1, h_4)
				(h_2, h_1)	(h_1, h_3)
				.	.

We describe a matching by a four entry vector that specifies the partner of each hospital in the order (h_1, h_2, h_3, h_4), so the matching $\mu = (s_1, s_2, s_3, s_4)$ implies that $\mu(h_1) = s_1$, $\mu(h_2) = s_2$ and so on. Note that according to Table 1, couples do not have responsive preferences, furthermore the set of stable matchings of this market is empty. Under these conditions, we establish the following result.

Claim 1 *Consider the market with couples described in Table 1, then there is no Nash equilibrium in pure strategies of the game* $G^{OA} = \left(S, \{H \cup \{s\}\}_{s\in S}, P^C\right)$ *induced by the* $\mathcal{OA}$.

Claim 1 shows that no well defined matching is attained in SPE for the market described in Table 1. Since $G^{OA} = \left(S, \{H \cup \{s\}\}_{s\in S}, P^C\right)$ is a strategic form game, there must exist at least one Nash equilibrium in mixed strategies. In the following result, we show that this problem disappears when couples' preferences are responsive[5].

[5]Recall that responsiveness is sufficient, but not necessary to assure the existence of stable matching. The result of Proposition 1 is even more general and holds for any market $\left(P^H, P^C\right)$ where $S\left(P^H, P^C\right) \neq \varnothing$.

Proposition 1 *Let $\left(P^H, P^C\right)$ be a market with couples where couples' preferences are responsive, then any stable matching of the market can be attained as a SPE outcome of the game induced by the $\mathcal{OA}$.*

This result follows the line of previous findings in the context of college admissions problems[6]. A natural question is whether only stable matchings are expected SPE outcomes of the $\mathcal{OA}$. The answer to this question is negative, as we show in the following example the mechanism may attain unstable matchings in SPE.

Example 1 *Consider a 4x4 market with couples with $c_1 = (s_1, s_2)$ and $c_2 = (s_3, s_4)$ and the following preferences,*

Table 2

Hospitals' Preferences				Couples' Preferences	
P^H				P^C	
h_1	h_2	h_3	h_4	$c_1 = (s_1, s_2)$	$c_2 = (s_3, s_4)$
s_2	s_3	s_1	s_2	(h_3, h_1)	(h_4, h_2)
s_3	s_4	s_3	s_3	(h_1, h_2)	(h_4, h_3)
s_4	s_1	s_4	s_1	(h_4, h_1)	(h_4, h_1)
s_1	s_2	s_2	s_4	(h_2, h_1)	(h_3, h_2)
$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	(h_3, h_2)	(h_3, h_4)
				(h_1, h_3)	(h_3, h_1)
				(h_4, h_2)	(h_1, h_2)
				(h_2, h_3)	(h_1, h_4)
				(h_3, h_4)	(h_1, h_3)
				(h_1, h_4)	(h_2, h_4)
				(h_4, h_3)	(h_2, h_1)
				(h_2, h_4)	(h_2, h_3)
				.	.

Couples' preferences in Table 2 are completed to be responsive. According to Proposition 1, stable matchings like $\mu_9 = (s_2, s_3, s_1, s_4)$ and $\mu_{11} = (s_2, s_4, s_1, s_3)$ can be supported in SPE.

Now consider that hospitals follow their dominant strategy, i.e. each $h \in H$ follows the decision rule $C_h(\cdot)$. Let m be a profile of students' messages that satisfies the following: $m(s_1) = h_1$, $m(s_2) = h_3$, $m(s_3) = h_4$ and $m(s_4) = h_2$. Clearly, the outcome of this profile is the matching, $\mu_5 = (s_1, s_4, s_2, s_3)$. We show that the profile of messages m is a Nash equilibrium in pure strategies of the game G^{OA}.

Note that neither s_3 nor s_4 deviate from m, since (h_4, h_2) is the top choice of the couple $c_2 = (s_3, s_4)$. For s_1 there is no $h_p \in H \cup \{u\}$ such that $(h_p, \mu_5(s_2)) \succ_{c_1} (\mu_5(s_1), \mu_5(s_2))$, hence there is no profitable deviation for s_1. Consider the case of s_2, note that only h_2 is a candidate for a profitable deviation, since $(\mu_5(s_1), h_2) \succ_{c_1} (\mu_5(s_1), \mu_5(s_2))$. However, if s_2 deviates with the alternative message $m'(s_2) = h_2$, the hospital h_2 (following the optimal rule $C_{h_2}(\cdot)$) will choose s_4. Hence, s_2 will be unmatched after deviating and $(\mu_5(s_1), \mu_5(s_2)) \succ_{c_1} (\mu_5(s_1), u)$ by responsiveness. So, there is no profitable deviation for any student. Then the matching $\mu_5 = (s_1, s_4, s_2, s_3)$ is a Nash equilibrium in pure strategies of G^{OA} and by construction a SPE outcome of the extensive form game Γ^{OA}. However, μ_5 is not stable since it is blocked by the coalition: $\{c_1 = (s_1, s_2), (h_3, h_1)\}$.

This feature of the problem contrasts with previous findings in college admissions problems. However, note that the kind of instability attained in equilibrium outcomes of the $\mathcal{OA}$ is very particular, since the matching $\mu_5 = (s_1, s_4, s_2, s_3)$ is only blocked by the coalition

[6]See, Alcalde, Pérez-Castrillo and Romero-Medina (1998), Alcalde and Romero-Medina (2000) and Triossi (2009).

$\{c_1 = (s_1, s_2), (h_3, h_1)\}$. Note that under the matching μ_5, we have that $\mu_5(s_1) = h_1$ and $\mu_5(s_2) = h_3$. Hence, the couple $c_1 = (s_1, s_2)$ and hospitals h_1 and h_3 block μ_5, because the members of the couple $c_1 = (s_1, s_2)$ are able to exchange their positions in a mutually profitable way for s_1, s_2, h_1 and h_3 to induce the matching $\mu_{11} = (s_2, s_4, s_1, s_3)$. In addition, if members of the couple $c_1 = (s_1, s_2)$ coordinate their strategies such that s_1 applies to h_3 and s_2 applies to h_1, the outcome of the $\mathcal{OA}$ would be the matching $\mu_{11} = (s_2, s_4, s_1, s_3)$, that is a SPE outcome since μ_{11} is stable. This implies that μ_5 is a SPE outcome of the $\mathcal{OA}$ because there is a coordination failure between the members of the couple c_1. In the following result, we prove that this feature of equilibrium outcomes is general and we show that only this kind of instability is possible in equilibrium.

Proposition 2 *Let (P^H, P^C) be a market with couples where couples' preferences are responsive, then any SPE outcome of the game induced by the $\mathcal{OA}$ is a matching that is either stable or blocked by some coalition of the form: $\{c = (s_k, s_l), (\mu(s_l), \mu(s_k))\}$.*

In the rest of this section, we show that the instability of SPE outcomes is compatible with the well known notion of pairwise stability. First of all, we introduce the formal definition of this concept,

Definition 6 *A matching μ is blocked by a pair (h_p, s_k) such that $h_p \in H \cup \{u\}$ and $s_k \in S$ if,*

1. $(h_p, \mu(s_l)) P_c (\mu(s_k), \mu(s_l))$ *and*
2. $s_k P_{h_p} \mu(h_p)$.

In a similar way, a matching μ is blocked by a pair (h_q, s_l) such that $h_q \in H \cup \{u\}$ and $s_l \in S$. A matching μ is **pairwise stable**, if it is individually rational and not blocked by any pair.

Note that the notion of pairwise stability is weaker than the usual notion of stability in markets with couples. Observe that a pairwise stable matching μ may be blocked by coalitions of the form $\{c = (s_k, s_l), (\mu(s_l), \mu(s_k))\}$. This observation makes possible to establish our main result.

Theorem 1 *Let (P^H, P^C) be a market where couples' preferences are responsive, then the $\mathcal{OA}$ implements in SPE the set of pairwise stable matchings of the market.*

The proof of Theorem 1 comes directly from Propositions 1 and 2, this result shows that the instability of SPE outcomes of the $\mathcal{OA}$ is not so strong. Only instabilities that come from coordination failures between members of couples are expected when agents face strategically the $\mathcal{OA}$. Furthermore, those instabilities are compatible with the usual notion of pairwise stability in matching problems. Note that no kind of coordination among agents is assumed along the paper, since usually it is very difficult to justify in strategic environments. However, the coordination of actions between members of a couple is very reasonable and natural in many economic environments. Then, it is not difficult to argue that only stable matchings will be reasonable equilibrium outcomes of the $\mathcal{OA}$.

4 Decentralized vs centralized mechanisms.

Our main result shows that when agents face strategically decentralized mechanisms, only pairwise stable matchings are expected equilibrium outcomes in markets with couples. This contrasts with the case of centralized mechanisms. One of the most known and analyzed matching procedures is the National Resident Matching Program[7] (NRMP). Klaus, Klijn and Massó (2007)

[7]The National Resident Matching Program (NRMP) is a centralized mechanism where is applied an algorithm to match hospitals and medical students in the USA. The purpose of the NRMP is matching hospitals and physician in a stable way, in this algorithm the presence of couples has been explicitly incorporated. See Roth, A. (1984) and Roth, A. (2008) to have a clear idea about the importance of the market of new physicians in the theory of market design.

analyze the new NRMP algorithm and show that it may attain unstable matchings when couples' preferences are responsive. To illustrate the problem, we consider the following example summarized in Table 3.

Table 3

Hospitals' Preferences				Couples' Preferences	
P^H				P^C	
h_1	h_2	h_3	h_4	$c_1 = (s_1, s_2)$	$c_2 = (s_3, s_4)$
s_2	s_2	s_2	s_2	(h_1, h_2)	(h_2, h_3)
s_3	s_3	s_3	s_3	(h_1, h_3)	(h_2, h_4)
s_1	s_1	s_1	s_1	(h_1, h_4)	(h_2, h_1)
s_4	s_4	s_4	s_4	(h_2, h_1)	(h_1, h_3)
$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	(h_2, h_3)	(h_1, h_4)
				(h_2, h_4)	(h_1, h_2)
				(h_3, h_1)	(h_3, h_4)
				(h_3, h_2)	(h_3, h_2)
				(h_3, h_4)	(h_3, h_1)
				(h_4, h_1)	(h_4, h_3)
				(h_4, h_2)	(h_4, h_2)
				(h_4, h_3)	(h_4, h_1)
				.	.

Couples' preferences are completed to be responsive, this assumption implies that there exists at least one stable matching. Klaus, Klijn and Massó (2007) apply that new NRMP algorithm to the previous example and find that this algorithm cycles over the unstable matching $\mu = (\emptyset, s_3, s_4, \emptyset)$. Note that μ is neither stable nor pairwise stable, since the pair (s_1, h_1) blocks μ. According to Theorem 1, μ cannot be supported by any SPE under the $\mathcal{OA}$. If the stability of outcomes measures the success of a matching mechanism, our main result shows that decentralized matching mechanisms work very well and better than centralized mechanisms for some instances of markets with couples.

5 Conclusion.

We show that a simple decentralized matching mechanism, like the $\mathcal{OA}$, implements in SPE the set of pairwise stable matchings of markets with couples. In contrast with the NRMP algorithm, only pairwise stable matchings are expected SPE outcomes of the $\mathcal{OA}$. Given the usual notion of stability in these setting, we show that the $\mathcal{OA}$ may attain unstable matchings in equilibrium. However, we also show that this instability is very particular and comes exclusively from coordination failures between members of couples.

6 Appendix: Proofs.

Claim 1 *Consider the market with couples described in Table 1, then there is no Nash equilibrium in pure strategies of the game* $G^{OA} = \left(S, \{H \cup \{s\}\}_{s\in S}, P^C\right)$ *induced by the* $\mathcal{OA}$.

Proof. Each profile of pure strategies yields a matching, hence it is enough to show that for every profile of messages there is a profitable deviation for at least one student.

Consider any matching ν, where there is at least one unmatched agent. This implies that there is at least one unmatched student and a hospital with an unfilled position, say s and h. Let m be any profile of messages that yields the matching ν, and let $m(s)$ be the message of the student s. Since $\nu(h) = \emptyset$ and all students are acceptable for hospitals, we know that $M(h) = \varnothing$. Hence, the alternative message $m'(s) = h$ is a profitable deviation for s, since the hospital h follows its dominant strategy and chooses $C_h(M'(h) \cup \emptyset) = s$ from $M'(h) = M(h) \cup \{s\}$. Then only matchings with no unmatched agents are candidates for equilibrium outcomes in pure strategies.

In the following list, we show the whole set of matchings with no unmatched agents. On the right of each matching, we show a student who has a profitable deviation given any profile of messages that yields each of these possible matchings:

Table 4

Matching	Profitable deviation	Matching	Profitable deviation
$\mu_1 = (s_1, s_2, s_3, s_4)$	$m'(s_4) = h_2$	$\mu_{13} = (s_3, s_1, s_2, s_4)$	$m'(s_4) = h_2$
$\mu_2 = (s_1, s_2, s_4, s_3)$	$m'(s_4) = h_2$	$\mu_{14} = (s_3, s_1, s_4, s_2)$	$m'(s_2) = h_3$
$\mu_3 = (s_1, s_3, s_2, s_4)$	$m'(s_2) = h_4$	$\mu_{15} = (s_3, s_2, s_1, s_4)$	$m'(s_2) = h_4$
$\mu_4 = (s_1, s_3, s_4, s_2)$	$m'(s_4) = h_1$	$\mu_{16} = (s_3, s_2, s_4, s_1)$	$m'(s_2) = h_3$
$\mu_5 = (s_1, s_4, s_2, s_3)$	$m'(s_2) = h_4$	$\mu_{17} = (s_3, s_4, s_1, s_2)$	$m'(s_1) = h_1$
$\mu_6 = (s_1, s_4, s_3, s_2)$	$m'(s_4) = h_1$	$\mu_{18} = (s_3, s_4, s_2, s_1)$	$m'(s_2) = h_1$
$\mu_7 = (s_2, s_1, s_3, s_4)$	$m'(s_4) = h_1$	$\mu_{19} = (s_4, s_1, s_2, s_3)$	$m'(s_4) = h_2$
$\mu_8 = (s_2, s_1, s_4, s_3)$	$m'(s_4) = h_2$	$\mu_{20} = (s_4, s_1, s_3, s_2)$	$m'(s_2) = h_3$
$\mu_9 = (s_2, s_3, s_1, s_4)$	$m'(s_2) = h_4$	$\mu_{21} = (s_4, s_2, s_1, s_3)$	$m'(s_2) = h_4$
$\mu_{10} = (s_2, s_3, s_4, s_1)$	$m'(s_4) = h_1$	$\mu_{22} = (s_4, s_2, s_3, s_1)$	$m'(s_2) = h_3$
$\mu_{11} = (s_2, s_4, s_1, s_3)$	$m'(s_2) = h_4$	$\mu_{23} = (s_4, s_3, s_1, s_2)$	$m'(s_3) = h_3$
$\mu_{12} = (s_2, s_4, s_3, s_1)$	$m'(s_4) = h_1$	$\mu_{24} = (s_4, s_3, s_2, s_1)$	$m'(s_4) = h_4$

Then there is no Nash equilibrium in pure strategies of this game. ■

Proposition 1 *Let* $\left(P^H, P^C\right)$ *be a market with couples where couples' preferences are responsive, then any stable matching of the market can be attained as a SPE outcome of the game induced by the* $\mathcal{OA}$.

Proof. Take any matching $\mu \in S\left(P^H, P^C\right)$, we know that there is at least one since couples' preferences are responsive. Consider a strategy profile such that:

1. $m(s) = \mu(s)$ for all $s \in S$; and

2. $J_h(\cdot) = C_h(\cdot)$ for all $h \in H$.

Given the strategy profile $(m, J) = \left(\{\mu(s)\}_{s\in S}, \{C_h(\cdot)\}_{h\in H}\right)$, the outcome of the $\mathcal{OA}$ is the matching $g^{OA}(m, J) = \mu$. We show that there is no profitable deviation for any agent. We know that $C_h(\cdot)$ is an optimal decision rule for each hospital, then no hospital has a profitable deviation.

Consider any student $s \in S$, since μ is a stable matching we know that it is individually rational, then the alternative message $m'(s) = u$ is not a profitable deviation for any $s \in S$ such that $\mu(s) \in H$.

Suppose that $s \in S$ sends a message $m'(s_k) = h_p$ such that $(h_p, \mu(s_l)) P_c (\mu(s_k), \mu(s_l))$. Given that no other agent deviates, we know that $M'(h) = M(h)$ for all $h \neq \mu(s_k), h_p$. In addition, $M'(\mu(s_k)) = \varnothing$ and $M'(h_p) = M(h_p) \cup \{s_k\}$. Since μ is stable and $h_p \in H$, we know that $\mu(h_p) P_h s_k$, then $C_{h_p}(M'(h_p)) = \mu(h_p)$. Let ν be the outcome matching of the mechanism given the individual deviation $m'(s_k)$ then, $\nu(s_k) = u$ and $\nu(s_l) = \mu(s_l)$. Since μ is individually rational, it follows that $(\mu(s_k), \mu(s_l)) R_c (u, \mu(s_l))$. Then, no students has a profitable deviation. ■

The following result is auxiliary to characterize the set of SPE outcomes of the $\mathcal{OA}$.

Lemma 1 *Suppose that couples preferences are responsive, then for each couple $c \in C$ and for all $h'_p, h'_q, h_p, h_q \in H \cup \{u\}$ such that $(h'_p, h'_q) \neq (h_q, h_p)$, if $(h'_p, h'_q) P_c (h_p, h_q)$ implies either $h'_p \succ_{s_k} h_p$ or $h'_q \succ_{s_l} h_q$.*

Proof. There are three cases:

1. Assume that $(h'_p, h'_q) \neq (h_q, h_p)$ and either $h'_p = h_q$ or $h'_q = h_p$. Consider without lost of generality (w.l.g.) that $h'_q = h_p$ and suppose that $(h'_p, h'_q) P_c (h_p, h_q)$ ($(h'_p, h_p) P_c (h_p, h_q)$ since $h'_q = h_p$). In contradiction, suppose that $h_p \succeq_{s_k} h'_p$ and $h_q \succeq_{s_l} h_p$. If $h_p = h'_p$ then $h'_p = u$ and $h_p \neq h_q$, hence $h_q \succ_{s_l} h_p$. By responsiveness, we know that $(h_p, h_q) P_c (h'_p, h_p)$, which is a contradiction. If $h_p \neq h'_p$ then $h_p \succ_{s_k} h'_p$ and by responsiveness $(h_p, h_q) P_c (h'_p, h_q)$ and $(h'_p, h_q) R_c (h'_p, h_p)$, hence $(h_p, h_q) P_c (h'_p, h_p)$ a contradiction.
2. Assume that $(h'_p, h'_q) P_c (h_p, h_q)$ and either $h'_p = h_p$ or $h'_q = h_q$. Suppose w.l.g. that $h'_q = h_q$, hence $h_q \succeq_{s_k} h'_q$. Consider in contradiction that $h_p \succ_{s_k} h'_p$, by responsiveness we have $(h_p, h_q) P_c (h'_p, h'_q)$, a contradiction.
3. Assume that $h'_p \neq h_q$, $h'_q \neq h_p$, $(h'_p, h'_q) \neq (h_q, h_p)$ and $(h'_p, h'_q) P_c (h_p, h_q)$. Suppose in contradiction that $h_p \succeq_{s_k} h'_p$ and $h_q \succeq_{s_l} h'_q$, note that both preferences have to be strict. Hence, by responsiveness $h_q \succ_{s_l} h'_q$ implies $(h'_p, h_q) P_c (h'_p, h'_q)$ and $h_p \succ_{s_k} h'_p$ implies $(h_p, h_q) P_c (h'_p, h_q)$. Hence $(h_p, h_q) P_c (h'_p, h'_q)$, a contradiction.

This completes the proof. ■

The previous lemma is useful to prove the following result.

Proposition 2 *Let (P^H, P^C) be a market with couples where couples' preferences are responsive, then any SPE outcome of the game induced by the $\mathcal{OA}$ is a matching that is either stable or blocked by some coalition of the form: $\{c = (s_k, s_l), (\mu(s_l), \mu(s_k))\}$.*

Proof. Suppose that the $\mathcal{OA}$ attains the unstable matching μ as a SPE outcome. Assume that μ is not individually rational: If $(u, \mu(s_l)) P_c (\mu(s_k), \mu(s_l))$ the student s_k has a profitable deviation with $m'(s_k) = u$. We have a similar case when $(\mu(s_k), u) P_c (\mu(s_k), \mu(s_l))$. Consider the third possibility $(u, u) P_c (\mu(s_k), \mu(s_l))$. We know that $(u, u) \neq (\mu(s_l), \mu(s_k))$, hence by Lemma 1 it is satisfied either $u \succ_{s_k} \mu(s_k)$ or $u \succ_{s_l} \mu(s_l)$. Assume w.l.g. that $u \succ_{s_k} \mu(s_k)$, by responsiveness $u \succ_{s_k} \mu(s_k)$ implies $(u, \mu(s_l)) P_c (\mu(s_k), \mu(s_l))$, hence $m'(s_k) = u$ is a profitable deviation for s_k. A contradiction, hence the matching μ has to be individually rational.

The previous argument implies that by assumption there has to exist at least one blocking coalition, say $\{c = (s_k, s_l), (h_p, h_q)\}$. There are three possible cases:

1. Assume either $h_p = \mu(s_k)$ or $h_q = \mu(s_l)$. Consider w.l.g. that $h_q = \mu(s_l)$ then $m'(s_k) = h_p$ is a profitable deviation for s_k. Since no other agent deviates if $h_p \in H$, then $M'(h_p) = M(h_p) \cup \{s_k\}$ and $\mu(h_p) \in M(h_p)$. This implies that the hospital h_p will choose the candidate $C_{h_p}(M'(h_p)) = s_k$ which confirms that $m'(s_k) = h_p$ is a profitable deviation for s_k. A contradiction.

2. Assume that $h_p \neq \mu(s_k)$, $h_q \neq \mu(s_l)$ and $(h_p, h_q) \neq (\mu(s_l), \mu(s_k))$. By Lemma 1 (case 3), we know that $(h_p, h_q) P_c (\mu(s_k), \mu(s_l))$ implies either $h_p \succ_{s_k} \mu(s_k)$ or $h_q \succ_{s_l} \mu(s_l)$. Suppose that $h_p \succ_{s_k} \mu(s_k)$, so by responsiveness $(h_p, \mu(s_l)) P_c (\mu(s_k), \mu(s_l))$. Hence, as in the previous case the message $m'(s_k) = h_p$ is a profitable deviation for s_k. A contradiction.

3. Assume that $(h_p, h_q) \neq (\mu(s_l), \mu(s_k))$ and either $h_p = \mu(s_l)$ or $h_q = \mu(s_k)$. By Lemma 1 (case 1), $(h_p, h_q) P_c (\mu(s_k), \mu(s_l))$ implies either $h_p \succ_{s_k} \mu(s_k)$ or $h_q \succ_{s_l} \mu(s_l)$. So, by responsiveness it is possible either $(h_p, \mu(s_l)) P_c (\mu(s_k), \mu(s_l))$ or $(\mu(s_k), h_q) P_c (\mu(s_k), \mu(s_l))$. As in previous cases, any of the students s_k or s_l has a profitable deviation. A contradiction.

There is only one more possibility, the blocking coalition: $\{c = (s_k, s_l), (\mu(s_l), \mu(s_k))\}$. However, we have already shown in Example 1 that it is easy to construct a profile of strategies that supports this kind of instability in SPE. This completes the proof. ■

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