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On Uncertainty and the WTA-WTP Gap

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Abstract

We correct an analysis by Isik (2004) regarding the effects of uncertainty on the WTA-WTP gap. Isik presents as his primary result a proposition that the introduction of uncertainty regarding environmental quality improvements causes WTA to increase and WTP to decrease by identical amounts relative to a certainty condition where $WTA=WTP$. These conclusions are incorrect. In fact, WTP may equal WTA even with uncertainty, and increases in the uncertainty of environmental quality improvements cause both WTA and WTP to fall. Finally, increases in uncertainty may either increase or decrease the WTA –WTP discrepancy.

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I. Introduction

A well-known feature of empirical studies in environmental and resource economics is that elicited measures of the maximum amount people indicate that they are willing to pay for a good (*WTP*) often differs substantially from the minimum amount they must be paid in order to forego it (the willingness to accept, *WTA*). A variety of explanations for this disparity have been proposed, including loss aversion (Thaler, 1980) and a limited degree of substitutability between the environmental good and other marketed goods (Hanemann, 1991). However, in a review of several empirical studies of the *WTP-WTA* gap, Horowitz and McConnell (2002) report that in addition to these other factors, uncertainty about the value of non-marketed products, such as environmental goods also likely contribute to the observed difference.

Isik (2004) attempts to systematically explore the relationship between uncertainty regarding an environmental quality improvement and the *WTP-WTA* gap. As his primary result he claims (his Proposition 1) that uncertainty causes the *WTA* to exceed *WTP* and that increases in uncertainty cause this gap to increase.¹ To prove this proposition Isik uses second order Taylor's series approximations of a risk averse agent's indirect utility function about several initial positions. Reasoning inappropriately from these approximations, Isik asserts (i) that in the absence of uncertainty $WTA=WTP$ if and only if the indirect utility function is linear in both income and the environmental improvement, and (ii) as uncertainty is introduced (or increased) *WTA* rises and *WTP* falls by identical amounts, making $WTA>WTP$. Both claims are importantly incorrect.

First, the *only if* portion of claim (i) is false. Consider as a simple counterexample the indirect utility function $V(y_0, q) = 1 - e^{-0.0005(y_0 + q)}$ where y denotes income and q an environmental quality index. This function is standard in the sense that it is strictly concave in both y and q , and as such is also nonlinear both arguments. Now let $y_0=5,000$, $q=2,000$ and consider a certain environmental improvement $\Delta = 300$. As can be readily verified, $WTP=WTA=\$300$.² Thus the indirect utility function can be nonlinear in both y and q and still yield $WTP=WTA$.

Still more problematically, Isik's claims in (ii) (i.e., his Proposition 1) are also false. In fact, for an agent who is risk averse in the environmental good, the introduction of uncertainty causes both *WTA* and *WTP* to fall. Further *WTA* need not exceed *WTP* and increases in uncertainty need not increase the *WTA-WTP* gap. This paper reveals the problems with these claims. In addition to pointing out the errors in his analysis, we establish as a general matter that increases in uncertainty causes both *WTA* and *WTP* to fall, and that the *WTA - WTP* gap can be expressed as a function of the *WTA* at two different income levels.

¹ Actually, Isik states the opposite (e.g., that uncertainty causes the compensating variation (*WTP*) to exceed the equivalent variation (*WTA*) and that increases in uncertainty further increase this discrepancy). However, as he purports to prove the opposite in his development, we assume that the text reflects an editing error.

² As articulated below in the text, *WTP* and *WTA* can be derived from the Hicksian welfare measures of compensating variation and equivalent variation. In this example, $WTP=300$ solves $V(y, q) = V(y - WTP, q + \Delta)$, and $WTA=300$ solves $V(y, q + \Delta) = V(y + WTA, q)$.

2. Isik's Development and Errors.

2.1 Assumptions and Notation. Following Isik, define an agent's utility function as $U(x, y, q)$ where x is a vector of market goods, q is a non-marketed environmental good and y is income. The agent solves $\text{Max}_x U(x, y, q)$ subject to $px = y$ where p is a vector of prices. The solution yields the vector of demand functions $x^* = x(p, y, q)$, which allow definition of the indirect utility function $V(p, y, q) = U(x^*, y, q)$. For notational ease, in what follows we suppress the price variable from the indirect utility function. Further, for specificity we define a reference level of income, $y = y_0$. Thus, we write indirect utility as $V(y_0, q)$.

Following Isik assume also that the marginal utilities of both the environmental good and income are positive, e.g. $V_q > 0$ and $V_y > 0$, that the agent has a linear utility of income, $V_{yy} = 0$, is risk averse in the environmental good $V_{qq} < 0$ and that y and q are complements in indirect utility, or $V_{yq} > 0$. Finally, let environmental quality improvement Δ be a random variable with mean $\bar{\Delta}$ and variance δ .

We calculate *WTP* and *WTA* from the Hicksian welfare measures of compensating variation (C) and equivalent variation (T), respectively (Freeman, 2003). Specifically compensating variation is derived from

$$V(y_0, q) = E[V(y_0 - C, q + \Delta)] \quad (1)$$

and equivalent variation from

$$V(y_0 + T, q) = E[V(y_0, q + \Delta)], \quad (2)$$

where E is the expectation operator defined over the necessarily random indirect utility.

*2.2 An Overview of Isik's Results.*³ To develop the compensating variation, C , Isik modifies (1) with the correct but unnecessarily cumbersome identity

$$E[V(y_0 - C, q + \Delta)] = V(y_0 - C - R^c, q + \bar{\Delta}), \quad (3)$$

where R^c is the additional risk premium that the agent is willing to pay to replace the random improvement Δ with its mean $\bar{\Delta}$, as a certainty. Taking a pair of second order Taylor series approximations for the indirect utility function at different reference points Isik approximates the compensating variation as

³ For a more complete development of material in the first part of this subsection see Isik (2004). Appendix A develops more fully the corrections to Isik's analysis (e.g., equations (4') and (6')).

$$C \cong \frac{\bar{\Delta}V_q + \frac{\bar{\Delta}^2}{2}V_{qq}}{V_y + \bar{\Delta}V_{qy}} + \frac{\delta V_{qq}}{2 V_y}. \quad (4)$$

Isik approximates the equivalent variation in a similar manner. First he modifies equation (2) by inserting a risk premium R^T in the left side of the equation and replaces the random Δ on the right side of equation (2) with its mean $\bar{\Delta}$, received as a certainty, to get

$$V(y_0 + T - R^T, q) = V(y_0, q + \bar{\Delta}). \quad (5)$$

Taking a pair of second order Taylor series approximations, again at different points, Isik approximates the equivalent variation as

$$T \cong \frac{\bar{\Delta}V_q + \frac{\bar{\Delta}^2}{2}V_{qq}}{V_y} - \frac{\delta V_{qq}}{2 V_y}. \quad (6)$$

Comparing (4) to (6) and recalling that $V_{qq} < 0$ and $V_y > 0$ Isik concludes that uncertainty drives a wedge between compensating variation (C) and equivalent variation (T), with the former decreasing and the latter increasing in a symmetric fashion as uncertainty increases.

These conclusions, however, are importantly incorrect because both C in (4) and T in (6) are miscalculated. In both (4) and (6), Isik fails to account for the differing reference points used in his sequential Taylor series approximations. In his development of (6) Isik further errs by incorrectly inserting risk premium R^T into the left hand side of equation (5): The risk premium is not a correction to T that an agent pays to achieve certainty in place of the random Δ . Once the agent sells the gamble for T , the uncertain Δ is irrelevant. Rather, the R^T is the amount the agent is willing to pay from the reference position $(y_0, q + \Delta)$ to replace the random Δ with its expected value $\bar{\Delta}$, as a certainty. Thus R^T must satisfy $E[V(y_0, q + \Delta)] = V(y_0 - R^T, q + \bar{\Delta})$. Making the appropriate adjustments, the corrected versions of (4) and (6) become

$$C \cong \frac{\bar{\Delta}V_q + \frac{\bar{\Delta}^2}{2}V_{qq}}{V_y + \bar{\Delta}V_{qy}} + \frac{\delta \hat{V}_{qq}}{2 \hat{V}_y} \quad (4')$$

and

$$T \cong \frac{\bar{\Delta}V_q + \frac{\bar{\Delta}^2}{2}V_{qq}}{V_y} + \frac{\delta \tilde{V}_{qq}}{2 \tilde{V}_y} \left(\frac{V_y + \bar{\Delta}V_{qy}}{V_y} \right) \quad (6')$$

where V , V_y , V_q and V_{qq} denote values of indirect utility and associated derivatives evaluated at the point (y_0, q) ; \hat{V}_y and \hat{V}_{qq} denote values of derivatives evaluated at point $(y_0 - C, q + \bar{\Delta})$ and; \tilde{V}_y and \tilde{V}_{qq} denote values of derivatives evaluated at $(y_0, q + \bar{\Delta})$.

Inspection of the corrected expressions for C and T in (4') and (6') allows two observations. First, comparing the corrected equivalent variation in (6') with Isik's formulation in (6), notice that the sign on the δ term switches from negative to positive with $V_y, \tilde{V}_y > 0, \tilde{V}_{qq} < 0$ and $V_{qy} > 0$, indicating that uncertainty causes the equivalent variation to fall rather than increase with uncertainty. Second, comparing across the corrected expressions for C and T in (4') and (6') observe that beyond the conclusion that uncertainty in q causes both compensating and equivalent variations to fall, little can be said. In particular, no conclusions may be drawn regarding the relative magnitude C and T . The corrected expression for compensating variation in (4') allows no insight into the absolute magnitude of C , since the right hand side is itself a function of C through \hat{V}_y and \hat{V}_{qq} . The risk adjustment terms \hat{V}_{qq}/\hat{V}_y (4') and $\tilde{V}_{qq}/\tilde{V}_y$ in (6') are evaluated at different points. Also, the risk adjustment term in (6') is weighted by interaction factor $(V_y + \Delta V_{qy})/V_y$ which is of indeterminate size.

3. Equivalent Variation, Risk and Improved Taylor Series Approximation

3.1 Equivalent Variation and Risk. Of the two preceding observations, the most general is that increasing risk causes the equivalent variation T as well as the compensating variation C to fall. We summarize the relationship between uncertainty, T and C with the following Proposition.

Proposition IA. Given uncertainty about an environmental quality improvement and an agent who is risk averse in the environmental good, an increase risk will reduce both the equivalent variation (T) and the compensating variation (C).

Proof: Denote an agent's indirect utility function by $V(y, q)$. Assume that $V_y > 0, V_q > 0$, and $V_{qq} < 0$ (e.g., strict concavity in q). Consider two prospects, each involving a favorable but uncertain change in environmental quality from an initial position q and with initial income at y_0 . Let Δ denote the uncertain change in the first prospect and z the uncertain change in the second prospect, where the *pdf* of z is a mean-preserving spread (MPS) of the *pdf* of Δ . The expected utility of the prospects are therefore $E[V(y_0, q + \Delta)]$ and $E[V(y_0, q + z)]$.

Uncertainty and the Equivalent Variation (T). Let $T(y_0, q + \Delta)$ denote the willingness to accept the loss of favorable prospect Δ , and $T(y_0, q + z)$ denote the willingness to accept the loss of the favorable prospect z . By definition, $E[V(y_0, q + \Delta)] = V(y_0 + T(y_0, q + \Delta), q)$ and $E[V(y_0, q + z)] = V(y_0 + T(y_0, q + z), q)$. Since z is a *MPS* of Δ , strict concavity of V with respect to q implies that $E[V(y_0, q + \Delta)] > E[V(y_0, q + z)]$ by Theorem 2 of Rothschild and Stiglitz (1970). Hence,

$$V(y_0 + T(y_0, q + \Delta), q) > V(y_0 + T(y_0, q + z), q).$$

By $V_y > 0$, $y_0 + T(y_0, q + \Delta) > y_0 + T(y_0, q + z)$ or $T(y_0, q + \Delta) > T(y_0, q + z)$, as asserted.

Uncertainty and the Compensating Variation (C). Let $C(y_0, q + \Delta)$ denote the willingness to pay for the favorable prospect Δ , and $C(y_0, q + z)$ denote the willingness to pay for the favorable prospect z . By definition $E[V(y_0 - C(y_0, q + \Delta), q + \Delta)] = V(y_0, q)$ and

$$E[V(y_0 - C(y_0, q + z), q + z)] = V(y_0, q) \quad (7)$$

By Theorem 2 of Rothschild and Stiglitz (1970), for any $0 < k < y_0$, $E[V(y_0 - k, q + \Delta)] > E[V(y_0 - k, q + z)]$. Thus,

$$E[V(y_0 - C(y_0, q + \Delta), q + \Delta)] > E[V(y_0 - C(y_0, q + \Delta), q + z)] \quad (8)$$

and by the favorability of z ,

$$E[V(y_0, q + z)] > V(y_0, q) = E[V(y_0 - C(y_0, q + \Delta), q + \Delta)]. \quad (9)$$

Combining (8) and (9) yields,

$$E[V(y_0, q + z)] > V(y_0, q) > E[V(y_0 - C(y_0, q + \Delta), q + z)]. \quad (10)$$

By the intermediate value theorem and $V_y > 0$, there exists a y^* , $y_0 > y^* > [y_0 - C(y_0, q + \Delta)]$, such that $E[V(y^*, q + z)] = V(y_0, q)$. Then by (7), $y^* = y_0 - C(y_0, q + z)$. Thus, $y_0 - C(y_0, q + z) > y_0 - C(y_0, q + \Delta)$, or $C(y_0, q + \Delta) > C(y_0, q + z)$ as asserted. \square

3.2 Improved Taylor Series Approximations of C and T. Although we can draw no general conclusions regarding the relative magnitude of C and T , the following more succinct development makes some progress in clarifying the relationship between the two variables in the presence of uncertainty regarding the level of an environmental improvement. We generate an expression for C , by making a second order Taylor series approximation to $V(y_0 - C, q + \Delta)$ about the point $V(y_0, q)$. Similarly, we generate an expression for T , by using a second order Taylor series approximation to $V(y_0 + T, q + \Delta)$ about the point $V(y_0, q)$. In each case we follow Isik's assumption that $V_{yy} = 0$. For C , the Taylor Series approximation for any particular Δ is

$$V(y_0 - C, q + \Delta) = V - CV_y + \Delta V_q - C\Delta V_{qy} + \frac{\Delta^2}{2} V_{qq} \quad (11)$$

where, as above, V, V_y, V_q, V_{qy} and V_{qq} denote the implicit function values and derivative values evaluated at (y_0, q) . Taking the expectation of both sides of (11) (noting that $E(\Delta^2) = \bar{\Delta}^2 + \delta$), and solving yields

$$C = \frac{\bar{\Delta}V_q + \frac{\bar{\Delta}^2}{2}V_{qq}}{V_y + \bar{\Delta}V_{qy}} + \frac{\delta}{2} \frac{V_{qq}}{V_y + \bar{\Delta}V_{qy}} \quad (12)$$

The equivalent variation T is approximated similarly, except that the correct defining equation is $EV(y_0, q + \Delta) = V(y_0 + T, q)$. Using a Taylor's second order approximation to $V(y_0, q + \Delta) = V([y_0 + T] - T, q + \Delta)$ about the point $V(y_0 + T, q)$ and solving yields

$$V([y_0 + T] - T, q + \Delta) = \tilde{V} - T\tilde{V}_y + \Delta\tilde{V}_q - T\tilde{V}_{qy} + \frac{\Delta^2}{2}\tilde{V}_{qq} \quad (13)$$

where $\tilde{V}, \tilde{V}_y, \tilde{V}_{qy}, \tilde{V}_{qq}$ and \tilde{V}_{qq} are evaluated at the point $V(y_0 + T, q)$. Taking the expectation of both sides, and again recalling that $E(\Delta^2) = \bar{\Delta}^2 + \delta$, (13) may be solved for T as

$$T = \frac{\bar{\Delta}\tilde{V}_q + \frac{\bar{\Delta}^2}{2}\tilde{V}_{qq}}{\tilde{V}_y + \bar{\Delta}\tilde{V}_{qy}} + \frac{\delta}{2} \frac{\tilde{V}_{qq}}{\tilde{V}_y + \bar{\Delta}\tilde{V}_{qy}} \quad (14)$$

Notice that C in (12) and T in (14) are identical in form and differ only in that their associated derivatives are evaluated at the different points, (y_0, q) and $(y_0 + T, q)$, respectively. Note further that if (14) is rewritten to give a second order approximation for T at the point $(y_0 - T(y_0, q), q)$, it will be identical to the second order approximation for C at (y_0, q) , consistent with a similar finding by Weber (2003). This last observation is important because it allows reduction of the comparison $T(y_0, q) - C(y_0, q)$ to an expression in T values only, via the comparison

$$T(y_0, q) - C(y_0, q) = T(y_0, q) - T(y_1, q) \quad (15)$$

where $y_1 = y_0 - T(y_0, q)$, which indicates that, at least up to a second order approximation (and with $V_{yy}=0$) the relationship between T and C is determined by the effect of changes in income on an agent's equivalent variation (WTA).

Looking at (14) notice that under Isik's rather restrictive assumptions regarding the indirect utility function T does increase with income (at least up to a second order approximation), meaning that $T > C$, as Isik asserts but fails to demonstrate. To see this, notice in the expression for T in (15) that only V_q changes with income (V_q moves directly with income since $V_{qy} > 0$). All other terms remain constant because $V_{yy} = 0$, and the higher order terms V_{qqy} and V_{qyy} are absent from a second order approximation). This conclusion, however, is specific to

his assumptions about the indirect utility function. For example, consider a change in the sign on the cross-partial term to $V_{qy} < 0$. In this case T moves inversely with income making $C > T$ (thus contradicting Isik's claim that his Proposition 1 does not require $V_{qy} > 0$ --see his note 1, p.3.).

As a simple example consider the utility function $V(y, q) = 100y + 100q - 0.05yq - 0.01q^2$. Let $y_0 = 800, q = 50$, and the uncertain environmental improvement be represented by the probability distribution $(\Delta_1, \Delta_2, prob[\Delta_1]) = (50, 100, .5)$. Over the relevant range, $V_y > 0, V_q > 0, V_{yy} = 0, V_{qq} < 0$ and $V_{qy} < 0$. Then $C = 46.5334$ and $T = 44.7436$ solve (1) and (2) respectively, making $C > T$.

Finally, Isik's claim that the *WTA-WTP* gap necessarily widens with introduction of uncertainty in the environmental quality improvement is false as well, even under his most restrictive set of assumptions. As a counterexample consider the utility function

$V(y, q) = 10y + 2yq - 0.002yq^2$, which has $V_y > 0, V_q > 0, V_{yy} = 0, V_{qq} < 0$, and $V_{qy} > 0$ over the

environmental quality interval $0 < q < 500$. Let $y_0 = 500, q = 110$, and the environmental improvement $\Delta = 30$ as a certainty. Then $C = 89.7129$ and $T = 109.3295$ solve (1) and (2) respectively, making $T - C = 19.6166$. Maintaining this structure, but replacing $\Delta = 30$ with a 'riskier' mean-preserving spread reflected in the probability distribution

$(\Delta_1, \Delta_2, prob[\Delta_1]) = (20, 40, .5)$, $C = 86.7470$ and $T = 104.9563$ solve (1) and (2) respectively,

making $T - C = 18.2093$. Thus the introduction of uncertainty may narrow rather than widen the *WTA-WTP* gap, contrary to Isik's claim. Notice again that both *WTP* and *WTA* fall with the introduction of uncertainty, but the *WTA-WTP* gap narrows here because *WTA* falls faster than *WTP*.

4. Conclusion

Empirical evidence suggests that uncertainty regarding the qualities of non-marketed goods may importantly affect the *WTP-WTA* gap, and we commend Isik for attempting to formalize this relationship. Isik, however, errs importantly in his development. Contrary to his claim, *WTP* may equal or exceed *WTA* even with uncertainty. Further, the introduction of uncertainty causes both *WTA* and *WTP* to fall and the *WTP-WTA* gap may decrease or increase with the introduction of uncertainty in an environmental quality improvement, even under Isik's strictest set of assumptions regarding the indirect utility function.

Finally, we observe that the use of Taylor series approximations offers a less than ideal method for isolating the relationship between compensating and equivalent variations when the contemplated changes in assets are not infinitesimal. The accuracy of such approximations are only assured in a neighborhood of the point at which its derivatives are evaluated. Regarding the use of second order approximations in representing individual behavior under risk, it should be noted that the widely accepted assumption of decreasing absolute risk aversion in income requires the third order derivative of indirect utility with respect to income to be positive.

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Appendix A. A More Complete Corrected Development of *WTP* and *WTA* Using the Approach Taken by Isik.

This appendix more fully develops the statements of *WTP* and *WTA* expressed in the text as equations (4') and (6').

A.1. Calculating WTP.

To derive *WTP* Isik takes a second order Taylor series approximation of the right hand side of (3) about the point (y_0, q) . Denoting $V(y_0, q)$ as V this yields

$$V(y_0 - C - R^c, q + \bar{\Delta}) \cong V - (C + R^c)V_y + \bar{\Delta}V_q - \bar{\Delta}(C + R^c)V_{qy} + \frac{\bar{\Delta}^2}{2}V_{qq}. \quad (\text{A1})$$

Using (1) in the text and solving for C yields

$$C \cong \frac{\bar{\Delta}V_q + \frac{\bar{\Delta}^2}{2}V_{qq}}{V_y + \bar{\Delta}V_{qy}} - R^c. \quad (\text{A2})$$

To eliminate R^c from (A2), Isik first obtains a Taylor series approximation of $V(y_0 - C, q + \Delta)$ for a given Δ about the point $(y_0 - C, q + \bar{\Delta})$ as

$$V(y_0 - C, q + \Delta) = \hat{V} + (\Delta - \bar{\Delta})\hat{V}_q + \frac{(\Delta - \bar{\Delta})^2}{2}\hat{V}_{qq} \quad (\text{A3})$$

Where \hat{V} denotes the reference indirect utility and derivative values evaluated at point $(y_0 - C, q + \bar{\Delta})$. He next obtains a Taylor series approximation of $V(y_0 - C - R^c, q + \bar{\Delta})$ about the point $(y_0 - C, q + \bar{\Delta})$ as

$$V(y_0 - C, q + \bar{\Delta}) = \hat{V} - R^c\hat{V}_y \quad (\text{A4})$$

Taking the expectation of (A3) with respect to Δ yields

$$E[V(y_0 - C, q + \Delta)] \cong \hat{V} + \frac{\delta}{2}\hat{V}_{qq}. \quad (\text{A5})$$

Using equation (3), we equate the right hand sides of (A4) and (A5) to obtain $R^c = -\frac{\delta}{2}\frac{\hat{V}_{qq}}{\hat{V}_y}$.

Substituting this expression for R^c into (A2) yields (4') in the text.

A.2 Calculating WTA.

Starting with the corrected insertion of risk premium R^T stated as (5') in the text, take a second order Taylor series approximation of the right hand side of the equation, $V(y_0 - R^T, q + \bar{\Delta})$, about the point (y_0, q) as

$$V(y_0 - R^T, q + \bar{\Delta}) \cong V - R^T V_y + \bar{\Delta} V_q + \frac{\bar{\Delta}^2}{2} V_{qq} - R^T \bar{\Delta} V_{qy} \quad (\text{A6})$$

Next, take second order Taylor series approximation of the left hand side of (5'), $V(y_0 + T, q)$, about (y_0, q) . This yields

$$V(y_0 + T, q) \cong V + T V_y \quad (\text{A7})$$

Substituting (A6) and (A7) into (5') and solving for T generates the following expression.

$$T \cong \frac{\bar{\Delta} V_q + \frac{\bar{\Delta}^2}{2} V_{qq}}{V_y} - \frac{(V_y + \bar{\Delta} V_{qy})}{V_y} R^T. \quad (\text{A8})$$

To eliminate R^T we use the corrected definition from the text above (5'),

$$V(y_0 - R^T, q + \bar{\Delta}) = E[V(y_0, q + \Delta)]. \quad (\text{A9})$$

Taking first a second order Taylor series approximation of $V(y_0 - R^T, q + \bar{\Delta})$ about the point $(y_0, q + \bar{\Delta})$, yields.

$$V(y_0 - R^T, q + \bar{\Delta}) \cong \tilde{V} - R^T \tilde{V}_y, \quad (\text{A10})$$

where \tilde{V} denotes values of indirect utility and the derivative evaluated at reference point $(y_0, q + \bar{\Delta})$. Next, take a second order Taylor series approximation of $V(y_0, q + \Delta)$ for a given Δ about the reference point $(y_0, q + \bar{\Delta})$ as

$$V(y_0, q + \Delta) \cong \tilde{V} + (\Delta - \bar{\Delta}) \tilde{V}_q + \frac{(\Delta - \bar{\Delta})^2}{2} \tilde{V}_{qq}$$

Then taking the expectation of both sides we have

$$E[V(y_0, q + \Delta)] \cong \tilde{V} + \frac{\delta}{2} \tilde{V}_{qq} \quad (\text{A11})$$

Replacing the left and right terms in (A9) with the right hand sides of (A10) and (A11), and solving yields $R^T = -\frac{\delta \tilde{V}_{qq}}{2 \tilde{V}_y}$. Replacing R^T with this expression in (A8) yields the expression (6') in the text.