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Egalitarianism, Variable Population and Monotonic Critical Level

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Abstract

This paper explores the introduction of a variable critical-level in a variable population context. We focus the attention on the Critical- Level Leximin criterion, a social evaluation procedure which compares two social states as follows: (i) It reproduces the Leximin criterion when applied to vectors of identical dimension and (ii) otherwise, it completes the smaller one with so many times a variable critical-level as to make the two vectors equal in size and applies the Leximin criterion again. We prove that the use of a strict monotonic critical-level leads to the intransitivity of the social evaluation criterion. This problem disappears when a weaker monotonicity condition is required.

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1 Introduction

This note deals with comparisons of vectors of real numbers of (possibly) different dimension. The context where these comparisons take place is the evaluation of social states with different population in a welfarist framework. Welfarist means that two social states are going to be compared by looking only at the utility level achieved by each member of the society in the social states at stake.

The social criterion historically attached to the early utilitarians is Classical (or Total) Utilitarianism. This criterion declares a social state better than another if the sum of the utilities of the individuals is higher in the former than in the latter.

Total utilitarianism has been seriously questioned when applied to a variable population setting, because it leads to Parfit's (1984) *repugnant conclusion* (in Arrhenius (2012) version): *For any perfectly equal population with very high welfare of each individual, there is a population with very low positive welfare which is better.*

There has been many attempts to overcome the repugnant conclusion. For instance, the Average Utilitarianism, which evaluates the social states by the average utility of their members, avoids the repugnant conclusion. Other social criteria like Ng's (1989) Theory X and Sider's (1991) Principle GV, which are compromise criteria between Total and Average Utilitarianism, avoid the repugnant conclusion as well.

Also within the utilitarian spirit, another proposal that avoids the repugnant conclusion is the Critical Level Utilitarianism, discussed and axiomatized by Blackorby et al. (2005). In their proposal, they introduce the notion of a critical level, a utility amount such that the addition to a given society of a new member with precisely that utility level, makes the society to be equally well-off. Then, the value of a social state is the summation of the differences between each individual's utility and a constant critical level.

The egalitarian perspective, as opposed to utilitarianism, is represented by some social criteria that give priority to the worst-off individual in the population. In particular, the Leximin criterion compares two social states by looking at the worst-off individuals in a lexicographic way: if the worst-off individuals in two social states are equally well, the criterion looks at the second worst-off individuals; and if the ties continue, it looks to the next worst-off individuals, up to the point in which a clear priority is reached.

The Leximin criterion avoids the repugnant conclusion but it has its own drawbacks. In particular, under the Leximin, any large population with a high level of well-being, u, is worse than a one-individual population with a utility slightly greater than the minimum of u. This fact is sometimes called the Reverse Repugnant Conclusion.

Again, the introduction of a constant critical level allows the Leximin Criterion to avoid this shortcoming. Blackorby et al. (1996) define the Constant Critical Level Leximin in the following way: in order to compare two populations of different dimension, we add to the one with the smaller number of individuals so many times a constant critical level as to make the two vectors equal in size and then the Leximin is applied to these equal-dimension vectors.

In a recent work, Asheim and Zubert (2012) propose the Rank-Discounted Utilitarianism as an attempt to fill out the gap between Critical Level Utilitarianism and their own version of the Constant Critical Level Leximin. Again, they use a constant critical level and their proposal satisfies most of the ethical principles that have been proposed in the literature, but not all (see Arrhenius (2000), (2012) for a very interesting impossibility theorem).

In this paper, we explore the consequences of the Leximin criterion together with a *variable* critical level. In our view, it is interesting to consider the case in which the critical level is not constant, but depends on the particular state of social affairs under consideration. First of all, it is very plausible that the critical level expressed in physical terms (consumption or income) may vary across different societies depending on physical factors such as their climate or their geographical situation. Second, it is also plausible that different societies may value consumption or income according to their different cultural or historical background. Suppose, for instance, the following immigration policy: The society would not accept any additional immigrants if their expected income (well-being) is going to be less than 50 per cent of the average income (well-being) of the existing population. In this case, social comparisons are made by using a variable critical level. For example, any critical level that relies on the arithmetic or geometric mean, weighted or not, is a variable critical level.

We will show that, when a property of Monotonicity is required, the use of a variable critical level may lead the Leximin to be intransitive. The property states that whenever some individual becomes strictly better-off, being the rest at least equally well, then the critical level of the society increases. Although this assumption fits well with the idea behind a variable critical level (a richer society welcomes new members at a minimum standard of living greater than a poorer one), it is totally incompatible with the use of the Leximin criterion. Finally, we present a possibility result showing that if we require a weaker version of monotonicity, then there exists a variable critical level that makes the Leximin to be transitive. Proofs of all the results are presented in the Appendix.

2 Notation and definitions

Before introducing variable population welfare criteria, some notation is needed. The number of people in some given society will be denoted by $n \in \mathbb{Z}_{++}$ (the set of positive integers). $\mathbb{R}$ ($\mathbb{R}_{++}$) stands for the set of real (positive real) numbers and $\mathbb{R}^n$ ($\mathbb{R}^n_{++}$) is the n-fold Cartesian product of $\mathbb{R}$ ($\mathbb{R}_{++}$). A vector $u = (u_1, u_2, .., u_n) \in \mathbb{R}^n$ is interpreted as a distribution of individual utilities, and it is assumed that these utilities are fully measurable and interpersonally full comparable. We assume, as it is usual in this literature, the existence of a utility level of *neutrality*. An individual utility level above neutrality means that her life, as a whole, is worth living. This neutrality level has been normalized, as usual, to zero. $\mathcal{R}$ stands for the union, for all $n \in \mathbb{Z}_{++}$, of $\mathbb{R}^n$, $\mathcal{R} \equiv \cup_n \mathbb{R}^n$. And for all $n \in \mathbb{Z}_{++}$, given $u, v \in \mathbb{R}^n$, $u \geq v, u > v, u >> v$ mean respectively: $u_i \geq v_i \; \forall i \in \{1, .., n\}$; $u \geq v$ and $u \neq v$; $u_i > v_i \; \forall i \in \{1, .., n\}$. For any $n \in \mathbb{Z}_{++}$ and $u \in \mathbb{R}^n$, u^* denotes a permutation of u such that $u_1^* \geq u_2^* \geq ... \geq u_n^*$. For all $u \in \mathcal{R}$, $dim(u)$ stands for the dimension of u, that is, the number of elements in vector u. Finally, given two societies with utility distributions $u \in \mathbb{R}^n$ and $v \in \mathbb{R}^m$, then $(u, v) \in \mathbb{R}^{n+m}$ denotes the utility distribution of a society obtained by the union of the previous two.

By means of a variable population social welfare criterion, distributions of utilities which are not necessarily of the same dimension can be compared. A social welfare criterion is a binary relation $\succsim$ on $\mathcal{R}$ with the usual interpretation: for all $x, y \in \mathcal{R}$, $x \succsim y$ means "x is at least as good as y from the social viewpoint". The indifference relation, $\sim$, is defined by $x \sim y \Leftrightarrow [x \succsim y$ and $y \succsim x]$, and the strict preference relation, $\succ$, by $x \succ y \Leftrightarrow [x \succsim y$ and $\neg(y \succsim x)]$, where $\neg$ denotes the logical negation. We assume that $\succsim$ is complete.

The present paper introduces a variable population social criterion based on the use of the Leximin criterion together with a *variable critical-level*.

Definition 1. *Given* $\succsim \subseteq \mathcal{R} \times \mathcal{R}$, *a* critical-level *associated to* $\succsim$ *is a function* $\varphi : \mathcal{R} \longrightarrow \mathbb{R}$ *such that for all* $u \in \mathcal{R}$, $(u, \varphi(u)) \sim u$. *If there exist* $u, v \in \mathcal{R}$ *such that* $u \neq v$ *and* $\varphi(u) \neq \varphi(v)$, *then we say that* φ *is a* variable critical-level function. *Otherwise we say that* φ *is a* constant critical-level.

A critical-level function provides, for each distribution of individual utilities, the utility that an additional person should enjoy in order to keep social welfare unchanged, given that the utilities of the existing population are unaffected. Next, we introduce the following property of some critical-level functions.

- **Strict Monotonicity (SMON):** $\forall n \in \mathbb{Z}_{++}$ and $\forall u, v \in \mathbb{R}^n$ such that $u > v$, $\varphi(u) > \varphi(v)$.

Strict Monotonicity calls for an increase of the critical-level whenever the utility distribution of a given society changes in such a way that no agent is worse-off and at least one agent is better-off. Some examples of critical-level functions satisfying these properties are the arithmetic and the geometric mean.

In this paper, we are interested in the extension of the Leximin criterion to a variable population context. First, we start presenting the classical Leximin criterion for a fixed population size.

Definition 2. *For* $n \in \mathbb{Z}_{++}$, *the* Leximin *criterion on* $\mathbb{R}^n$, $\geq_L^n$, *is defined as follows: for all* $u, v \in \mathbb{R}^n$,

$$u \geq_L^n v \Leftrightarrow \nexists k \in \{1, \ldots, n\} \mid u_i^* = v_i^* \; \forall i \in \{k+1, \ldots, n\} \wedge v_k^* > u_k^*.$$

Let $\succsim_L$ be the union of $\geq_L^n$ for all $n \in \mathbb{Z}_{++}$. Given $m, n \in \mathbb{Z}_{++}$ with $m > n$, any $u \in \mathbb{R}^n$ and any $r \in \mathbb{R}$, $u(r)^m$ stands for the m-dimensional extension of u by means of r. That is:

$$[u(r)^m]_i = \begin{cases} u_i & \text{if } i \in \{1, \ldots, n\} \\ r & \text{if } i \in \{n+1, \ldots, m\}. \end{cases}$$

There are several possible extensions of the Leximin criterion to the variable population case. We propose the following:

Definition 3. *For any function $\varphi : \mathcal{R} \longrightarrow \mathbb{R}$, $\succ_L^{\varphi} \subseteq \mathcal{R} \times \mathcal{R}$ is the $\varphi-$*critical-level egalitarian criterion *if for all $u \in \mathbb{R}^n$, $v \in \mathbb{R}^m$ such that $n \leq m$,*

$$u \succsim_L^{\varphi} v \Leftrightarrow u(\varphi(u))^m \succsim_L v$$

In spite of this seemingly reasonable extension of the Leximin criterion, the use of a strict monotonic critical-level may lead to intransitivities, as the next example shows:

Example 2.1. *Consider the utility distributions corresponding to three different societies:*

$$u = (9, 9, 9, 9, 2) \in \mathbb{R}^5; \;\; v = (10, 10, 10, 2) \in \mathbb{R}^4 \text{ and } w = (20, 9.5, 2) \in \mathbb{R}^3.$$

Suppose that the critical-level function *is the arithmetic mean, μ. Then, given that $\mu(v) = 8$ and $\mu(w) = 10.5$,*
$u \succ_L^{\mu} v$ since $(9, 9, 9, 9, 2) >_L^5 (10, 10, 10, 8, 2)$,
$v \succ_L^{\mu} w$ since $(10, 10, 10, 2) >_L^4 (20, 10.5, 9.5, 2)$,
and $w \succ_L^{\mu} u$ since $(20, 10.5, 10.5, 9.5, 2) >_L^5 (9, 9, 9, 9, 2)$.

From the previous example, some natural questions immediately arise. How can we ensure that a $\varphi-$critical-level egalitarian criterion is transitive? Is SMON of a critical level function φ compatible with a transitive $\varphi-$critical-level egalitarian criterion? The following section analyzes these questions.

3 The meaning of transitivity: an impossibility result

According to Example 1 presented above, we cannot guarantee the transitivity of $\succ_L^{\varphi}$, which is an intuitive candidate to extend the Leximin criterion to a variable population framework. In this section, we will characterize the transitivity of a $\varphi-$critical-level egalitarian criterion by means of two properties.

The first condition, Number of Entrants Independence, requires the critical-level of a given society to be the same as the critical-level of the society obtained by adding a member to the initial one with a utility amount corresponding to its critical-level. The idea behind this property has some relationship with the "Self-identity" property in Yager and Rybalov (1997).

- **Number of Entrants Independence (NEI)**: $\forall u \in \mathcal{R}$, $\varphi(u, \varphi(u)) = \varphi(u)$.

The second property, Critical-Level Consistency, demands the ranking between two utility distributions to be independent of the incorporation, to each society, of a further person with the utility amount corresponding to its critical-level.

- **Critical-Level Consistency (CLC)**: Given $\succsim \subseteq \mathcal{R} \times \mathcal{R}$ and $u, v \in \mathcal{R}$, $u \succsim v \Leftrightarrow (u, \varphi(u)) \succsim (v, \varphi(v))$.

Now, we can prove that these two properties are necessary and sufficient to ensure transitivity of $\succsim_L^\varphi$.

Proposition 3.1. *$\succsim_L^\varphi$ is transitive if and only if φ satisfies NEI and $\succsim_L^\varphi$ satisfies CLC.*

Proposition 3.1 shows necessary and sufficient conditions for the $\varphi-$critical-level egalitarian criterion to be transitive. The following result shows that the use of the Leximin criterion together with a strict monotonic critical level leads to intransitive social choices.

Theorem 3.2. *$\nexists \varphi : \mathcal{R} \longrightarrow \mathbb{R}$ satisfying SMON and such that $\succsim_L^\varphi$ is transitive.*

4 Weakening strict monotonicity

In this section we propose a weaker property of monotonicity that shares the flavor of our initial proposal. As we will prove, this weaker version allows for a transitive extension of the Leximin criterion.

- **Monotonicity (MON):** $\forall n \in \mathbb{Z}_{++}, \forall u, v \in \mathbb{R}^n$, if $u >> v$ then $\varphi(u) > \varphi(v)$.

MON calls for an increase of the critical-level whenever the utility distribution of a given society changes in such a way that each individual is strictly better-off.

The following result shows that when only this weaker version of monotonicity is imposed, we can find a meaningful family of variable critical-level functions that are compatible with the transitivity of the φ-critical-level egalitarian criterion. These functions are based on the utility of the worst-off individual in the society in such a way that they accomplish well with the ethical foundations of the Leximin criterion, but avoid the unsuitable situation in which a new individual with a negative utility is welcome in the society.

Proposition 4.1. *Let $c \in \mathbb{R}_{++}$ be a strictly positive real number. Then, let $\varphi_c : \mathcal{R} \to \mathbb{R}_+$ be defined as follows: $\forall u \in \mathcal{R}$,*

$$\varphi_c(u) = \begin{cases} \frac{c}{1+|\min\{u_i\}|} & \textit{if} \min\{u_i\} \leq 0 \\ \\ c + \min\{u_i\} & \textit{if} \min\{u_i\} \geq 0. \end{cases}$$

Then, φ_c satisfies MON and $\succsim_L^{\varphi_c}$ is transitive.

5 Conclusions

Theorem 3.2 shows that SMON is incompatible with a transitive extension of the Leximin. MON is a weaker version of Monotonicity that allows for the existence of transitive Leximin extensions. We have shown a particular family of critical-level functions that satisfy MON and produces a transitive Leximin extension, but there can be some others. A characterization of the family of variable critical-level functions that satisfies MON and allows for a transitive Leximin extension is a topic for further research.

Appendix

Proof of Proposition 3.1

It is not difficult to prove that if $\succsim_L^\varphi$ is transitive, then φ satisfies NEI and $\succsim_L^\varphi$ satisfies CLC.[1] The proof of the sufficient part is established in two steps:
(i) If φ satisfies NEI and $\succsim_L^\varphi$ satisfies CLC, then $\succsim_L^\varphi$ satisfies Extended Critical-Level Consistency (ECLC), that is, given $u, v \in \mathcal{R}$, $u \succsim_L^\varphi v \Leftrightarrow (u, 1^n \cdot \varphi(u)) \succsim_L^\varphi (v, 1^n \cdot \varphi(v))$.

Let $u, v \in \mathcal{R}$ such that $u \succsim_L^\varphi v$ and let $n \in \mathbb{Z}_{++}$. By applying CLC we get that $u \succsim_L^\varphi v \Leftrightarrow (u, \varphi(u)) \succsim_L^\varphi (v, \varphi(v))$. Given that φ satisfies NEI, the application of CLC on $(u, \varphi(u))$ and $(v, \varphi(v))$ lead to $(u, \varphi(u)) \succsim_L^\varphi (v, \varphi(v)) \Leftrightarrow (u, \varphi(u), \varphi(u)) \succsim_L^\varphi (v, \varphi(v), \varphi(v))$. Therefore, $u \succsim_L^\varphi v \Leftrightarrow (u, \varphi(u), \varphi(u)) \succsim_L^\varphi (v, \varphi(v), \varphi(v))$. Repeating this reasoning $(n-2)$ times we obtain the desired result.
(ii) If $\succsim_L^\varphi$ satisfies ECLC and φ satisfies NEI, then $\succsim_L^\varphi$ is transitive. Let $u, v, w \in \mathcal{R}$ such that $u \succsim_L^\varphi v$, $v \succsim_L^\varphi w$, $dim(u) = r$, $dim(v) = s$ and $dim(w) = t$ with $max\{r, s, t\} = p$. By applying ECLC, NEI and taking into account the definition of $\succsim_L^\varphi$, we have that $u \succsim_L^\varphi v \Leftrightarrow (u, 1^{p-max\{r,s\}} \cdot \varphi(u)) \succsim_L (v, 1^{p-max\{r,s\}} \cdot \varphi(v)) \Leftrightarrow (u, 1^{p-r} \cdot \varphi(u)) \succsim_L (v, 1^{p-s} \cdot \varphi(v))$. By using the same reasoning, we get $v \succsim_L^\varphi w \Leftrightarrow (v, 1^{p-s} \cdot \varphi(v)) \succsim_L (w, 1^{p-t} \cdot \varphi(w))$. Given that $\succsim_L$ is transitive, $(u, 1^{p-r} \cdot \varphi(u)) \succsim_L (w, 1^{p-t} \cdot \varphi(w))$. Now, by definition, $(u, 1^{p-max\{r,t\}} \cdot \varphi(u)) \succsim_L^\varphi (w, 1^{p-max\{r,t\}} \cdot \varphi(w))$. Finally, by applying once more ECLC, we obtain $u \succsim_L^\varphi w$ and, therefore, $\succsim_L^\varphi$ is transitive.

Proof of Theorem 3.2

In order to prove the theorem, we need the following lemmata.

Lemma 5.1. *If $\succsim_L^\varphi$ is transitive, then $\forall u \in \mathcal{R}$, and for every permutation $\pi : \{1, \ldots, dim(u)\} \to \{1, \ldots, dim(u)\}$, $\varphi(u) = \varphi(\pi(u))$.*

[1] The proof is available from the authors upon request.

Proof. By definition of $\succsim_L^{\varphi}$, we have that $\forall u \in \mathcal{R}$, for any permutation $\pi : \{1,\dots,dim(u)\} \to \{1,\dots,dim(u)\}$, $u \sim_L^{\varphi} \pi(u)$. Moreover, we know by Proposition 3.1 that transitivity of $\succsim_L^{\varphi}$ implies CLC. Therefore, $(u,\varphi(u)) \sim_L^{\varphi} (\pi(u),\varphi(\pi(u)))$. Given that $\succsim_L$ is antisymmetric, we get $\varphi(u) = \varphi(\pi(u))$. ■

Lemma 5.2. *$\nexists \varphi : \mathcal{R} \to \mathbb{R}$ satisfying SMON such that $\succsim_L^{\varphi}$ is transitive and $\varphi(u) \geq u_2^* \; \forall u \in \mathbb{R}^n$ such that $n \geq 2$.*

Proof. Suppose that $\exists \varphi : \mathcal{R} \to \mathbb{R}$ satisfying the requirements. Let $u \in \mathcal{R}$. We know, by Proposition 3.1, that transitivity of $\succsim_L^{\varphi}$ implies NEI. Therefore, $\varphi(u,\varphi(u),\varphi(u)) = \varphi(u)$. Consider $v = (u_1 - 1, u_2, \dots, u_n, \varphi(u), \varphi(u))$. Then, $\varphi(v) \geq v_2^*$ and, by construction, $v_2^* \geq \varphi(u)$. Therefore, $\varphi(v) \geq \varphi(u)$. Now, since $v < (u,\varphi(u),\varphi(u))$, by applying SMON we get $\varphi(v) < \varphi(u)$, which contradicts the previous inequality. ■

Now, we are ready to prove the theorem. Suppose that $\exists \varphi : \mathcal{R} \to \mathbb{R}$ satisfying SMON and such that $\succsim_L^{\varphi}$ is transitive. By Lemma 5.2, $\exists n \geq 2$, and $u \in \mathbb{R}^n$ such that $\varphi(u) < u_2^*$. Considering u^*, by Lemma 5.1, $\varphi(u^*) = \varphi(u)$. Now, construct a family of vectors $v(k,x) = (u_1^* + k, u_2^* - x, u_3^*, .., u_n^*)$ for $k \geq 0$ and $x \in [0, u_2^* - \varphi(u)) \equiv [0,b)$. There are three possibilities:

(i) $\exists x \in [0,b)$, $y \in (x,b)$ and $k > 0$ such that $\varphi(v(0,x)) < \varphi(v(k,y))$. On the one hand, by definition of $\succsim_L^{\varphi}, v(0,x) \succ_L^{\varphi} v(k,y)$ because $[(v(0,x)]_2 = u_2^* - x > u_2^* - y = [(v(k,y)]_2$ and $[(v(0,x)]_j = [v(k,y)]_j \; \forall j > 2$. On the other hand, and taking into account that, by hypothesis, $\varphi(v(0,x)) < \varphi(v(k,y))$ and, by SMON, $\varphi(v(0,x)) \leq \varphi(v(0,0)) \equiv \varphi(u) < u_2^* - y$, the introduction of the critical levels will change the ordination; that is, $(v(0,x), \varphi(v(0,x))) \prec_L^{\varphi} (v(k,y), \varphi(v(k,y)))$. But this fact contradicts CLC and, by Proposition 3.1, transitivity.

(ii) $\exists x \in [0,b)$, $y \in (x,b)$ and $k > 0$ such that $\varphi(v(0,x)) = \varphi(v(k,y))$. By taking $z = \frac{x+y}{2}$, we are in case (i) since $\varphi(v(0,x)) < \varphi(v(k,z))$.

(iii) $\forall x \in [0,b)$, $\forall y \in (x,b)$ and $\forall k > 0$, $\varphi(v(0,x)) > \varphi(v(k,y))$. In this case we define the correspondence $F_{k^*} : [0,b) \to\to \mathbb{R}$ such that $\forall z \in [0,b)$, and a fixed $k^* > 0$, $F_{k^*}(z) = [\varphi(v(0,z)), \varphi(v(k^*,z))]$. Notice that, by SMON, F_{k^*} is always multivalued; that is, $\forall z \in [0,b)$, $F_{k^*}(z) = [a,b]$ with $a < b$. Moreover, taking into account the hypothesis $\varphi(v(0,x)) > \varphi(v(k,y))$ we get that $\forall x,y \in [0,b)$, $x \neq y$, $F_{k^*}(x) \cap F_{k^*}(y) = \emptyset$. But, since the number of disjoint intervals in $\mathbb{R}$ is countable and $[0,b)$ is not, our conclusion is absurd.

Proof of Proposition 4.1

It is straightforward to prove that φ_c satisfies MON for any $c \in \mathcal{R}_{++}$.

Now, in order to prove that $\succsim_L^{\varphi_c}$ is transitive, by Proposition 3.1 it is enough to prove that $\succsim_L^{\varphi_c}$ satisfies NEI and CLC.

Firstly, we will show that φ_c satisfies NEI. For any $n \in \mathbb{Z}_{++}$, take $u \in \mathbb{R}^n$. If $min\{u_i\} \geq 0$ then $\varphi_c(u) > min\{u_i\}$, and given that $(u, min\{u_i\}) \in \mathbb{R}_{++}^{n+1}$ and that

$min\{u_i\} = min\{u_i, \varphi_c(u)\}$, $\varphi_c(u, min\{u_i\}) = \varphi_c(u)$. Similarly, if $min\{u_i\} \leq c$ we also have that $\varphi_c(u) > min\{u_i\}$. Then, $min\{u_i\} = min\{u_i, \varphi_c(u)\}$ and, therefore, $\varphi_c(u, min\{u_i\}) = \varphi_c(u)$.

In order to prove that $\succsim_L^{\varphi_c}$ satisfies CLC, it is convenient to take into account that, for any $n \in \mathbb{Z}_{++}$, the Leximin criterion on $\mathbb{R}^n$, $\geq_L$, satisfies the two following properties:

- *Independence* ($In_{\mathbb{R}^n}$): $\forall u, v \in \mathbb{R}^n$, $\forall x \in \mathbb{R}$, $u \geq_L^n v \Leftrightarrow (u, x) \geq_L^{n+1} (v, x)$.
- *Monotonicity* ($Mon_{\mathbb{R}^n}$): $\forall u, v \in \mathbb{R}^n$, if $u > v$, then $u >_L^n v$.

Then, consider $u \in \mathbb{R}^p$, $v \in \mathbb{R}^q$. Assume that $u \sim_L^{\varphi_c} v$. If $p = q$ then $u^* = v^*$ and $u^* =_L^q v^*$. Given that $u^* = v^*$, we know that $\varphi_c(u) = \varphi_c(v)$, and by $In_{\mathbb{R}^n}$ that, $(u, \varphi_c(u))^* =_L^{q+1} (v, \varphi_c(v))^*$. Therefore $(u, \varphi_c(u)) \sim_L^{\varphi_c} (v, \varphi_c(v))$. We will assume now that $u \sim_L^{\varphi_c} v$ and that, without loss of generality, $p < q$. Then $(u, 1^{(q-p)} \cdot \varphi_c(u))^* = v^*$, thus $\varphi_c(u, 1^{(q-p)} \cdot \varphi_c(u)) = \varphi_c(v)$, which by NEI is equal to $\varphi_c(u)$. Then we can apply again $In_{\mathbb{R}^n}$ to obtain that $(u, 1^{(q-p+1)} \cdot \varphi_c(u))^* =_L^{p+1} (v, \varphi_c(v))^*$, which, by the definition of $\succsim_L^{\varphi_c}$ is what is required for $(u, \varphi_c(u)) \sim_L^{\varphi_c} (v, \varphi_c(v))$.

Assume now that $u \succ_L^{\varphi_c} v$. We have to prove that $(u, \varphi_c(u)) \succ_L^{\varphi_c} (v, \varphi_c(v))$. We will distinguish three cases: (1): $p = q$, (2): $p < q$, and (3): $p > q$.

Case 1: $u \succ_L^{\varphi_c} v$ and $p = q$. Then $u \succ_L^{\varphi_c} v$ implies $u^* >_L^p v^*$. This implies that $\min\{u\} \geq \min\{v\}$ and therefore, $\varphi_c(u) \geq \varphi_c(v)$. If $\varphi_c(u) = \varphi_c(v)$ then, by $In_{\mathbb{R}^n}$ we have that $(u, \varphi_c(u))^* >_L^{p+1} (v, \varphi_c(v))^*$, and therefore $(u, \varphi_c(u)) \succ_L^{\varphi_c} (v, \varphi_c(v))$ as required. If $\varphi_c(u) > \varphi_c(v)$, again by $In_{\mathbb{R}^n}$ we have that $(u, \varphi_c(u))^* >_L^{p+1} (v, \varphi_c(u))^*$, and by $Mon_{\mathbb{R}^n}$, $(v, \varphi_c(u))^* >_L^{p+1} (v, \varphi_c(v))^*$. Therefore, by transitivity of $>_L^{p+1}$, $(u, \varphi_c(u)) \succ_L^{\varphi_c} (v, \varphi_c(v))$ as required.

Case 2: $u \succ_L^{\varphi_c} v$ and $p < q$. Then $(u, 1^{(q-p)} \cdot \varphi_c(u))^* >_L^q v^*$. This implies that $\varphi_c(u, 1^{(q-p)} \cdot \varphi_c(u)) \geq \varphi_c(v)$. Also, by NEI we know that $\varphi_c(u, 1^{(q-p)} \cdot \varphi_c(u)) = \varphi_c(u)$. Then, we have that $\varphi_c(u) \geq \varphi_c(v)$. If $\varphi_c(u) = \varphi_c(v)$ then, by $In_{\mathbb{R}^n}$, $(u, 1^{(q-p+1)} \cdot \varphi_c(u))^* >_L^{q+1} (v, \varphi_c(u))^*$, which by the definition of $\succsim_L^{\varphi_c}$ is what is required for $(u, \varphi_c(u)) \succ_L^{\varphi_c} (v, \varphi_c(v))$. If $\varphi_c(u) > \varphi_c(v)$, again by $In_{\mathbb{R}^n}$, we have that $(u, 1^{(q-p+1)} \cdot \varphi_c(u))^* >_L^{q+1} (v, \varphi_c(u))^*$, and given that $\varphi_c(u) > \varphi_c(v)$, by $Mon_{\mathbb{R}^n}$ we have that $(v, \varphi_c(u))^* >_L^{q+1} (v, \varphi_c(v))^*$. Therefore, by transitivity of $>_L^{q+1}$, $(u, 1^{(q-p+1)} \cdot \varphi_c(u))^* >_L^{q+1} (v, \varphi_c(v))^*$, which by the definition of $\succsim_L^{\varphi_c}$ is what is required for $(u, \varphi_c(u)) \succ_L^{\varphi_c} (v, \varphi_c(v))$.

Case 3: $u \succ_L^{\varphi_c} v$ and $p > q$. Then $(u)^* >_L^q (v, 1^{(p-q)} \cdot \varphi_c(v))^*$. This implies that $\varphi_c(u) \geq \varphi_c(v, , 1^{(p-q)} \cdot \varphi_c(v)$. By NEI we know that $\varphi_c(v, 1^{(p-q)} \cdot \varphi_c(v)) = \varphi_c(v)$, therefore $\varphi_c(u) \geq \varphi_c(v)$. Proceeding analogously to Case 2, either if $\varphi_c(u) = \varphi_c(v)$ or if $\varphi_c(u) > \varphi_c(v)$ we obtain $(u, \varphi_c(u)) \succ_L^{\varphi_c} (v, \varphi_c(v))$.

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