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On the tacit collusion equilibria of a dynamic duopoly investment game

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Abstract

This note extends the characterization of simultaneous investment (tacit collusion) equilibria in Boyer, Lasserre and Moreaux (2012). Tacit collusion equilibria may or may not exist, and when they do may involve either finite time investments (type 1) or infinite delay (type 2). The relationship between equilibria and common demand forms is not immediately apparent. We provide the full necessary and sufficient conditions for existence. A simple condition on demand primitives is derived that determines the type of equilibria. Common demand forms are then shown to illustrate both finite-time and infinite-delay tacit collusion.

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1 Introduction

In a recent paper, Boyer, Lasserre, and Moreaux (henceforth BLM) study the possible investment paths in a continuous time noncooperative Cournot duopoly. Firms face market demand development uncertainty and may acquire lumpy capacity units at any point in time. Their work integrates both the more recent "real options" methodology and a timing game *à la* Fudenberg and Tirole (1985). The authors find that, when firms are capacity constrained, preemptive equilibria always exist, but also simultaneous equilibria in which both firms delay investing over a finite period, or never invest, may arise, depending on payoffs. Since there are no binding agreements in this framework, the simultaneous investment equilibria are habitually qualified as tacitly collusive.

This note extends the work of BLM (2012) by fully characterizing the conditions under which one or the other type of (Pareto superior) collusive outcome may obtain in equilibrium. These conditions are generally intricate, but we unveil a simple criterion to discriminate between finite or infinite delay in collusive investment choices that relates directly to the structure of demand. The easy applicability of our conditions is illustrated by examples. Using common demand forms we show that, in the linear demand case, in all collusive equilibria firms abstain from investing forever, and only by adding a curvature parameter do both types of equilibria arise.

For the note to be self-contained, in Section 2 we briefly present the model in the same notation as the original paper. The conditions for tacit collusion are provided in Section 3. Examples follow in Section 4.

2 The Model

We begin by describing (a case of) BLM's model and essential results.[1] Two firms compete in quantities in a market with inverse demand $P_t = Y_t D(X_t)$ where X_t is total quantity, $D(X_t)$ is positive, strictly decreasing, and strictly concave, and the shock Y_t follows a geometric Brownian motion, $dY_t = \alpha Y_t dt + \sigma Y_t dZ_t$, with $Y_0 > 0$, $\alpha > 0$ (growth rate), $\sigma > 0$ (volatility), and where $(Z_t)_{t\geq 0}$ is a standard Wiener process.[2] The timing of the game is as follows: 1) given the realiza-

[1] BLM describe investment choices over the entire "tree" for firms that begin with no initial capacity and may make multiple investments; our focus is on a subgame where both firms' capacities are near the Cournot output.

[2] The geometric brownian motion is derived from $Y_t = Y_0 \exp\left[\left(\alpha - \frac{1}{2}\sigma^2\right)t + \sigma Z_t\right]$ by using Itô's lemma. For the equation of motion to describe a market in expansion, it is assumed that $\alpha > \frac{\sigma^2}{2}$.

tion of Y_t, and existing capital stocks, each firm chooses to invest a number of "lumpy" capacity units; 2) given capacity units, each firm selects an output level under capacity constraint; 3) given output levels, market price is determined according to the inverse demand function.

Production is costless, so the optimal (unique and stable) per-period Cournot output x^c of each firm without capacity constraints is time-independent. Both firms are initially capacity constrained with capital stock $k \in \mathbb{N} \backslash \{0\}$ and each of them may relax the constraint by investing in one additional unit. The end of the investment game is near, in that $x^c - 1 \leq k < x^c$. Firms decide non-cooperatively (contracts are ruled out) and without commitment when to invest in an additional unit of capacity at cost I. Let $l = k + 1$. When a firm has i units of capacity while its rival has j units, its instantaneous variable profit is denoted as $Y_t \pi_{ij}$. Initially, with capacity k, both firms earn $Y_t \pi_{kk} = Y_t D(2k) k$. When they both have capacity l, they may sell x^c, so that $Y_t \pi_{ll} = Y_t D(2x^c) x^c$. Note that $\pi_{lk} > \pi_{kk} > \pi_{kl}$, and $\pi_{lk} > \pi_{ll} > \pi_{kl}$.

Letting $y \equiv Y_t$ stand for the current demand shock, BLM establish that the value of a firm f that invests immediately while its rival $-f$ invests at the threshold $y_{kl} > y$ is:

$$L_{kk}(y, y_{kl}) = \frac{\pi_{lk}}{r - \alpha} y - I + \left(\frac{y}{y_{kl}}\right)^{\beta} \frac{\pi_{ll} - \pi_{lk}}{r - \alpha} y_{kl},$$

where $\beta = \frac{1}{2} - \frac{\alpha}{\sigma^2} + \left[\left(\frac{1}{2} - \frac{\alpha}{\sigma^2}\right)^2 + \frac{2r}{\sigma^2}\right]^{\frac{1}{2}} > 1$, and $r > \alpha$ is a constant interest rate.[3]

A firm's value from investing as a follower at y_{kl} if its rival invests immediately at $y < y_{kl}$ is:

$$F_{kk}(y, y_{kl}) = \frac{\pi_{kl}}{r - \alpha} y + \left(\frac{y}{y_{kl}}\right)^{\beta} \left(\frac{\pi_{ll} - \pi_{kl}}{r - \alpha} y_{kl} - I\right).$$

Each firm may benefit from the growing demand by relaxing the capacity constraint before its rival. In the absence of commitment vis-à-vis investment choices, competition for the lead position results in a preemption equilibrium, in which one of the two firms – with equiprobability – invests before its rival. In this case a Markov Perfect Equilibrium (MPE) is determined by two investment triggers $y_{kk}^p < y_{kl}^*$. The trigger y_{kk}^p denotes the preemption threshold. It is defined implicitly as the smaller root of the equation $L_{kk}\left(y_{kk}^p, y_{kl}^*\right) = F_{kk}\left(y_{kk}^p, y_{kl}^*\right)$, where the larger root, $y_{kl}^* = \frac{\beta}{\beta - 1} \frac{r - \alpha}{\pi_{ll} - \pi_{kl}} I = \arg\max_{y_{kl}} F_{kk}(y, y_{kl})$, is the optimal follower threshold. Before y reaches y_{kk}^p, both firms earn $Y_t \pi_{kk}$. Between y_{kk}^p and y_{kl}^*, the leading firm f earns $Y_t \pi_{lk}$, while firm $-f$ earns $Y_t \pi_{kl}$. Both firms earn $Y_t \pi_{ll}$ when $y \geq y_{kl}^*$.

[3] The expression of β is standard. See Dixit and Pindyck ((1994), pp. 140-144) for a detailed exposition of the steps that lead to it.

Firms may also coordinate investment choices. The value, measured at y, if both firms invest simultaneously at the (possibly infinite) joint investment threshold $\bar{y}_{kk} > y$ is:

$$S_{kk}(y, \bar{y}_{kk}) = \frac{\pi_{kk}}{r-\alpha} y + \left(\frac{y}{\bar{y}_{kk}}\right)^{\beta} \left(\frac{\pi_{ll} - \pi_{kk}}{r-\alpha} \bar{y}_{kk} - I\right).$$

With the privately optimal simultaneous investment trigger y^s_{kk}, firms' payoff is $S^* \equiv S_{kk}(y, y^s_{kk})$.

3 Conditions for Tacit Collusion

Preemption equilibrium investment triggers $\left(y^p_{kk}, y^*_{kl}\right)$ always exist, but as BLM show, either simultaneous investments at a finite or infinite y^s_{kk} may constitute an MPE, that is Pareto dominant. We follow convention by assuming that when several equilibria exist, the Pareto dominant one is the most reasonable one to expect. As in BLM, we refer to coordination by the firms on simultaneous investment as tacit collusion. Formally:

Proposition 1 *(BLM, Prop. 5) Suppose $Y_0 \leq y^p_{kk}$:*

1. *A necessary and sufficient condition for a simultaneous MPE to exist is:*

$$S^* \geq L(y, y^*_{kl}), \ \forall y \leq y^*_{kl}. \tag{1}$$

If this inequality is strict, there exists a continuum of simultaneous MPEs. From the firms' point of view, these MPEs are Pareto ranked.

2. *The Pareto optimal simultaneous investment threshold is either $y^s_{kk} = \frac{\beta}{\beta-1}\frac{r-\alpha}{\pi_{ll}-\pi_{kk}} I = \arg\max_{y_{kk}} S(y, y_{kk})$ if $\pi_{ll} > \pi_{kk}$ (type 1 collusion), or infinite otherwise (type 2 collusion).*

The existence of the tacit collusion equilibrium therefore depends on the function S^*. Note that although (1) is very general as it does not refer explicitly to instantaneous profits, it may be expressed differently depending on the comparison of π_{kk} with π_{ll}, since:

$$S^* = \begin{cases} \frac{\pi_{kk}}{r-\alpha} y + \left(\frac{y}{y^s_{kk}}\right)^{\beta} \left(\frac{\pi_{ll}-\pi_{kk}}{r-\alpha} y^s_{kk} - I\right) & \text{if } \pi_{ll} > \pi_{kk}; \\ \frac{\pi_{kk}}{r-\alpha} y & \text{otherwise.} \end{cases} \tag{2}$$

With some algebra, the necessary and sufficient condition (1) may be expressed as follows:

Proposition 2 *(Necessary and sufficient conditions for collusive equilibria) Suppose* $Y_0 \le y^p_{kk}$:

1. *A type 1 collusion equilibrium exists if (BLM, Prop. 6) and only if* $\pi_{ll} > \pi_{kk}$ *and:*

$$(\pi_{lk} - \pi_{kk})^{\beta} \le (\pi_{lk} - \pi_{kk}) (\pi_{ll} - \pi_{kk})^{\beta-1} + (\beta - 1) (\pi_{lk} - \pi_{ll}) (\pi_{ll} - \pi_{kl})^{\beta-1} . \tag{3}$$

2. *A type 2 collusion equilibrium exists if and only if* $\pi_{ll} \le \pi_{kk}$ *and:*

$$(\pi_{lk} - \pi_{kk})^{\beta} \le \beta (\pi_{lk} - \pi_{ll}) (\pi_{ll} - \pi_{kl})^{\beta-1} . \tag{4}$$

Proof. (see Appendix) ■

BLM state the sufficient condition (3), but necessity is more elaborate to establish. In addition, Proposition 2 provides a necessary and sufficient condition for type 2 collusion.

Conditions (3) and (4) do not have obvious economic interpretations, but on the other hand we can remark that, when a Pareto optimal tacit collusion equilibrium exists, its nature hinges on the sign of the difference $\pi_{ll} - \pi_{kk}$, and this difference can be related simply to the demand primitive. We establish that the sign of this difference actually depends on the straightforward comparison of the initial stock of capital, k, with a critical level of output, x^*, which is the unique quantity strictly lower than x^c satisfying $D(2x^*) x^* = D(2x^c)x^c$. Existence and uniqueness of x^* result from the strict concavity of $D(X_t)$. On the basis of this, we can offer a characterization of the type of Pareto optimal tacit collusion equilibria in the model, as follows.

Proposition 3 *(Discrimination of collusive equilibria) Suppose* $Y_0 \le y^p_{kk}$. *The Pareto optimal collusive equilibrium is of type 1 (type 2) if and only if:*

$$x^* > (\le) k. \tag{5}$$

Proof. ($\Leftarrow$) For all $\bar{x} < x^*$ the definition of x^* implies $D(2\bar{x}) \bar{x} < D(2x^*)x^* = D(2x^c)x^c$. Then pick $\bar{x} = k$, to obtain $\pi_{kk} < \pi_{ll}$.

($\Rightarrow$) We have $X \equiv x_f + x_{-f}$. As $D(X) x_f$ is strictly concave, $D(2k) k < D(2x^c)x^c$ implies either $k > x^c$, which is ruled out by assumption, or (exclusively so) $k < x^*$. ■

This result completes the analysis in BLM by identifying an easy to use criterion that determines the type of collusion. The intuition is very clear. If the capacity constraint is very severe ($x^* > k$) in that firms' instantaneous joint profit is less than in the unconstrained Cournot case, it pays to invest more in a collusion equilibrium. Otherwise firms find it profitable to stop investing in order to earn superior profits at each point in time forever. Remark that $k = \lceil x^c \rceil - \mathbf{1}_{\mathbb{N}}(x^c)$, by assumption, so that the condition $k < x^*$ is equivalent to comparing the integer component of x^c, minus 1 only if x^c is natural, with x^*. The greater the pre-installed capacity k relative to Cournot output, the greater the likelihood of infinite delay.

BLM focus on type 1 equilibria, but type 2 equilibria are also noteworthy from an industrial organization perspective. When $x^* \leq k$ and (4) hold, despite instantaneous Cournot competition in output, the simultaneous equilibrium in investment choices mimics the kind of collusive quantity restriction that may emerge in a repeated game setting. This link between the dynamic investment model and product market outcomes provides a justification of BLM's use of the tacit collusion terminology.

4 Examples

We now study the applicability of Propositions 2 and 3 to different demand functions. To begin with, consider the common linear specification.

Example 1 *Suppose that $P(X) = a - bX$, with $a, b > 0$.*

Firm f's profit function is $\pi_f(x_f, x_{-f}) = (a - b(x_f + x_{-f}))\, x_f$. For type 1 collusion to occur, we know from Proposition 3 that $x^* > k$ is necessary. In this linear setup, it is easy to check that this condition is incompatible with the model's main assumptions, namely that the game is near its end, that is $x^c - 1 \leq k$, and that capacity units are lumpy, so that $k \geq 1$.

Claim 4 *In the case of linear demand, collusive equilibria are always of type 2.*

Proof. By Proposition 3, type 1 collusion arises if $x^* > k$. With $P(X) = a - bX$, we have $x^c = \frac{1}{3}\frac{a}{b}$ and it is direct to compute $x^* = \frac{1}{6}\frac{a}{b}$. By assumption, the lumpy pre-installed capacity must satisfy $k \geq 1$, and the near end condition imposes $x^c - 1 \leq k$. This latter condition implies that $\frac{1}{3}\frac{a}{b} \leq k + 1 \leq 2$, so $\frac{1}{6}\frac{a}{b} \leq 1$, hence $x^* \leq k$. ■

It follows that the linear specification is limited as an illustration of BLM's full analysis of tacitly collusive investment decisions. However, by considering a broader class of demand functions, we can illustrate all the possible cases of Propositions 2 and 3.

Example 2 *Suppose that* $P(X) = a - bX^{\delta}$, $\delta > 0$, *and let* $a = 4^{\delta}\left(1 + \frac{\delta}{2}\right)b$.

Consider the case where $k = 1$, $l = 2$, as in BLM.[4] Here the specific choice of a implies that $x^c = 2$, and allows us to focus on the role of the curvature parameter δ. By means of Propositions 2 and 3 above, we can completely characterize in terms of the parameters β and δ the collusive equilibria for this family of demand functions. To begin with, by Proposition 3:

Claim 5 The collusive equilibrium, if it exists, is of type 1 (type 2) if and only if $\delta > (\leq) 1$.

Proof. Simple computation establishes that $\pi(x^c, x^c) = \left(a - b4^{\delta}\right)2 = \delta 4^{\delta} b$. The critical output x^* that solves $\pi(x^*, x^*) = \pi(x^c, x^c)$ satisfies $(a - 2^{\delta} b x^{*\delta})x^* = \delta 4^{\delta} b$. Substituting $4^{\delta}\left(1 + \frac{\delta}{2}\right)b$ for a, we get x^* as the lower root of:

$$f_{\delta}(z) \equiv -2^{\delta} z^{\delta+1} + 4^{\delta}\left(1 + \frac{\delta}{2}\right)z - \delta 4^{\delta},$$

where $f_{\delta}(z)$ is concave in z over $\mathbb{R}_+$ and has $x^c = 2$ as its upper root. If $f_{\delta}(1) < 0$, then $x^* > k$ and collusion is of type 1 (conversely, if $f_{\delta}(1) \geq 0$, $x^* \leq k$ and collusion is of type 2). It is then sufficient to identify the roots of:

$$f_{\delta}(1) \equiv 2^{\delta}\left(1 - \frac{\delta}{2}\right) - 1,$$

where $f_{\delta}(1)$ is a concave function of δ over $\mathbb{R}_+$. The two roots are $\delta = 0$ (which is non admissible) and $\delta = 1$. For $\delta > 1$ (≤ 1), $f_{\delta}(1) < 0$ (≥ 0) and collusion is type 1 (type 2). ■

It is interesting to observe that the linear demand form of Example 1 exactly constitutes a limiting case of type 2 collusion with the specification that $\delta = 1$ in Example 2. Here the type 1 collusion that BLM focus on may occur only if demand satisfies their assumption of strict concavity, that is $\delta > 1$.

[4]Note that, although this demand does not satisfy the strict concavity assumption made by BLM for $\delta \leq 1$, the proofs of our propositions remain valid with this specification.

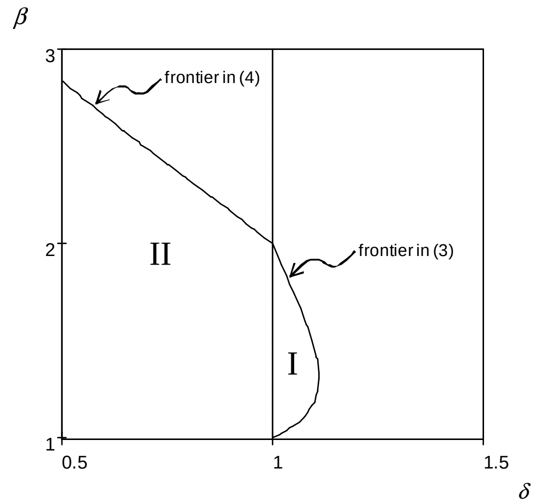

Figure 1: In Example 2, there is type 1 collusion in region I (points on the frontier $\delta = 1$, that correspond to the linear case, are excluded). There is type 2 collusion in region II.

The two conditions of Proposition 2 are checked by directly calculating the profits π_{11}, π_{12}, π_{21}, and π_{22}. As a result, we are able to numerically partition the parameter space in (β, δ) (the magnitude of b has no impact on the relative profits at different investment levels). The results are plotted in Figure 1 for the values $\beta \in (1, 3]$ and $\delta \in (1/2, 3/2]$ where all four possible scenarios (type 1/type 2 collusion, existence/nonexistence) arise.

References

[1] Boyer, M., P. Lasserre, and M. Moreaux (2012) "A Dynamic Duopoly Investment Game without Commitment under Uncertain Market Expansion" *International Journal of Industrial Organization* **30**, 663-681.

[2] Dixit, A. K., and R. S. Pindyck (1994) *Investment under Uncertainty*, Princeton University Press: Princeton.

[3] Fudenberg, D., and J. Tirole (1985) "Preemption and Rent Equalization in the Adoption of New Technology" *Review of Economic Studies* **52**, 383-401.

Appendix

Before proving Proposition 2, we establish the following lemma:[5]

Lemma 1 Under the assumptions of the model, if $\pi_{ll} > \pi_{kk}$, then $\pi_{lk} + \pi_{kl} > \pi_{ll} + \pi_{kk}$.

Proof. Define $\pi\left(x^f, x^{-f}\right) \equiv D\left(x^f + x^{-f}\right)x^f$, and let $\widetilde{x} = \arg\max_{x\leq l} \pi(x,k)$. Then, $\pi_{lk} + \pi_{kl} = \pi(\widetilde{x},k) + \pi(k,\widetilde{x}) \equiv \Pi(\widetilde{x}+k)$, where $\Pi(X)$ denotes industry profit. Let X^m denote the monopoly quantity. First, suppose that $\widetilde{x}+k < X^m$. Because $\pi\left(x^f, x^{-f}\right)$ is strictly submodular (we have $D' < 0$), $\pi_{ll} + \pi_{kk} = \pi(x^c,x^c) + \pi(k,k) < \pi(x^c,k) + \pi(k,x^c) = \Pi(x^c+k)$, and since $x^c+k \leq \widetilde{x}+k$ and industry profit is increasing to the left of X^m, we have $\pi_{ll}+\pi_{kk} < \Pi(x^c+k) \leq \Pi(\widetilde{x}+k)$. Otherwise, suppose that $\widetilde{x}+k \geq X^m$. Let $x^*(k) = \arg\max_{x\in\mathbb{R}_+} \pi(x,k)$, where $x^*(k)$ is the unconstrained best-response to k. Because $\left|\frac{\partial x^{f*}}{\partial x^{-f}}\right| < 1$ it must be that $x^*(k) + k < 2x^c$. Since $\widetilde{x} \leq x^*(k)$ we have $\widetilde{x}+k < 2x^c$. As $\widetilde{x}+k \geq X^m$, industry revenue is decreasing to the right of $\widetilde{x}+k$, so $\Pi(\widetilde{x}+k) \geq \Pi(2x^c)$. Finally, $\Pi(2x^c) = 2\pi(x^c,x^c) > \pi(x^c,x^c) + \pi(k,k)$ when $\pi_{ll} > \pi_{kk}$. Therefore $\Pi(\widetilde{x}+k) > \pi(x^c,x^c) + \pi(k,k) = \pi_{ll} + \pi_{kk}$. ■

Proof of Proposition 2

(1) By Proposition 1, and (2), a collusive equilibrium exists if and only if $S^* - L(y, y^*_{kl}) \geq 0$, $\forall y \leq y^*_{kl}$, where $S^* - L(y, y^*_{kl})$ is written as:

$$\frac{\pi_{kk}-\pi_{lk}}{r-\alpha}y + I + \left(\frac{y}{y^s_{kk}}\right)^{\beta}\left(\frac{\pi_{ll}-\pi_{kk}}{r-\alpha}y^s_{kk} - I\right) + \left(\frac{y}{y^*_{kl}}\right)^{\beta}\frac{\pi_{lk}-\pi_{ll}}{r-\alpha}y^*_{kl},$$

with $y^s_{kk} = \frac{\beta}{\beta-1}\frac{r-\alpha}{\pi_{ll}-\pi_{kk}}I$ and $y^*_{kl} = \frac{\beta}{\beta-1}\frac{r-\alpha}{\pi_{ll}-\pi_{kl}}I$. The function $S^* - L(y, y^*_{kl})$ is convex in y, strictly positive and decreasing at 0 (BLM). It is non-negative on the interval $[0, y^*_{kl}]$ if and only if either (i) it has a non-positive derivative and a non-negative value at y^*_{kl}, or (ii) its minimum value over $\mathbb{R}_+$ is non-negative. We know from BLM that the latter condition holds if and only if (3) holds. We show that the former condition cannot hold. To see that, compute the derivative of $S^* - L(y, y^*_{kl})$ w.r.t. y, that is:

$$-\frac{\pi_{lk}-\pi_{kk}}{r-\alpha} + \beta\left(\frac{y}{y^s_{kk}}\right)^{\beta-1}\left(\frac{\pi_{ll}-\pi_{kk}}{r-\alpha} - \frac{I}{y^s_{kk}}\right) + \beta\left(\frac{y}{y^*_{kl}}\right)^{\beta-1}\frac{\pi_{lk}-\pi_{ll}}{r-\alpha}.$$

[5] The same reasoning also holds for the potentially convex demand in Example 2.

Evaluated at y^*_{kl}, this gives:

$$\frac{(\beta-1)(\pi_{lk}-\pi_{ll})+\pi_{kk}-\pi_{ll}}{r-\alpha}+\frac{\pi_{ll}-\pi_{kk}}{r-\alpha}\left(\frac{\pi_{ll}-\pi_{kk}}{\pi_{ll}-\pi_{kl}}\right)^{\beta-1},$$

which is non-positive if:

$$(\pi_{ll}-\pi_{kk})^{\beta}\leq(\pi_{ll}-\pi_{kl})^{\beta-1}\left[\beta\pi_{ll}-\pi_{kk}-(\beta-1)\pi_{lk}\right].$$

The value of $S^*-L(y,y^*_{kl})$ at y^*_{kl} is:

$$-\frac{\beta}{\beta-1}\frac{\pi_{lk}-\pi_{kk}}{\pi_{ll}-\pi_{kl}}I+I+\frac{1}{\beta-1}I\left(\frac{\pi_{ll}-\pi_{kk}}{\pi_{ll}-\pi_{kl}}\right)^{\beta}+\frac{\beta}{\beta-1}\frac{\pi_{lk}-\pi_{ll}}{\pi_{ll}-\pi_{kl}}I,$$

which is non-negative if:

$$(\pi_{ll}-\pi_{kl})^{\beta-1}\left[\pi_{ll}-\beta\pi_{kk}+(\beta-1)\pi_{kl}\right]\leq(\pi_{ll}-\pi_{kk})^{\beta}.$$

Therefore, for a non-positive derivative and a non-negative value at y^*_{kl}, the following two conditions must hold jointly:

$$\begin{cases}(\pi_{ll}-\pi_{kk})^{\beta}\leq(\pi_{ll}-\pi_{kl})^{\beta-1}\left[\beta\pi_{ll}-\pi_{kk}-(\beta-1)\pi_{lk}\right]\\(\pi_{ll}-\pi_{kl})^{\beta-1}\left[\pi_{ll}-\beta\pi_{kk}+(\beta-1)\pi_{kl}\right]\leq(\pi_{ll}-\pi_{kk})^{\beta}\end{cases}.$$

However, this implies that:

$$\pi_{lk}+\pi_{kl}\leq\pi_{ll}+\pi_{kk},$$

violating Lemma 1. Hence $S^*-L(y,y^*_{kl})$ cannot have both a non-positive derivative and a non-negative value at y^*_{kl}. Consequently, it is non-negative on the interval $[0,y^*_{kl}]$ if and only (3) holds.

(2) In this case, $S^*-L(y,y^*_{kl})$ is written as:

$$-\frac{\pi_{lk}-\pi_{kk}}{r-\alpha}y+I+\left(\frac{y}{y^*_{kl}}\right)^{\beta}\frac{\pi_{lk}-\pi_{ll}}{r-\alpha}y^*_{kl},$$

with still $y^*_{kl}=\frac{\beta}{\beta-1}\frac{r-\alpha}{\pi_{ll}-\pi_{kl}}I$, and the derivative of $S^*-L(y,y^*_{kl})$ w.r.t. y is:

$$-\frac{\pi_{lk}-\pi_{kk}}{r-\alpha}+\beta\left(\frac{y}{y^*_{kl}}\right)^{\beta-1}\frac{\pi_{lk}-\pi_{ll}}{r-\alpha}.$$

As above, $S^* - L\left(y, y_{kl}^*\right)$ is a convex function of y that is strictly positive and strictly decreasing at the origin. Let us now study the behavior of $S^* - L\left(y, y_{kl}^*\right)$ at y_{kl}^*. The derivative at y_{kl}^* has the sign of:

$$(\beta - 1)\,\pi_{lk} + \pi_{kk} - \pi_{ll} > 0.$$

It follows that the minimum of $S^* - L\left(y, y_{kl}^*\right)$ always lies in $[0, y_{kl}^*)$. Note incidentally that $S^* - L\left(y, y_{kl}^*\right)$ is always positive at y_{kl}^*. A tacit collusion equilibrium therefore exists if and only if the value of $S^* - L\left(y, y_{kl}^*\right)$ at this minimum in non-negative. The derivative is zero when $y = \left(\frac{1}{\beta}\frac{\pi_{lk}-\pi_{kk}}{\pi_{lk}-\pi_{ll}}\right)^{\frac{1}{\beta-1}} y_{kl}^*$. The minimized value is then:

$$-\frac{\beta}{\beta-1}\frac{\pi_{lk}-\pi_{kk}}{\pi_{ll}-\pi_{kl}} I \left(\frac{1}{\beta}\frac{\pi_{lk}-\pi_{kk}}{\pi_{lk}-\pi_{ll}}\right)^{\frac{1}{\beta-1}} + \frac{1}{\beta-1}\frac{\pi_{lk}-\pi_{kk}}{\pi_{ll}-\pi_{kl}} I \left(\frac{1}{\beta}\frac{\pi_{lk}-\pi_{kk}}{\pi_{lk}-\pi_{ll}}\right)^{\frac{1}{\beta-1}} + I,$$

which is non-negative if and only if:

$$(1-\beta)\left(\frac{1}{\beta}\frac{\pi_{lk}-\pi_{kk}}{\pi_{lk}-\pi_{ll}}\right)^{\frac{1}{\beta-1}} + (\beta-1)\,\frac{\pi_{ll}-\pi_{kl}}{\pi_{lk}-\pi_{kk}} \geq 0,$$

or equivalently $\beta\left(\pi_{lk} - \pi_{ll}\right)\left(\pi_{ll} - \pi_{kl}\right)^{\beta-1} \geq \left(\pi_{lk} - \pi_{kk}\right)^{\beta}$. ■