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Punishment versus Reward in All-pay Contests with Perfect Information

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Abstract

We study when costly punishment induces higher expected effort than prizes in all-pay contests with perfect information. Punishment outperforms rewards if the number of players in the contest is large enough or if the principal can easily administer effective punishment. If the marginal cost of punishment is equal to the marginal cost of reward, then punishment induces more effort in all symmetric contests.

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1 Introduction

We show conditions such that costly punishment results in higher expected effort than prizes in all-pay contests with perfect information. In general, if the number of players is large enough or the marginal cost of punishment (to the players) is high enough, punishment induces more effort than reward. We derive specific results for the cases of symmetric and asymmetric contests.

First, in symmetric contests, we find a necessary and sufficient condition for punishing any number of the worst agents to induce more effort than using any number of symmetric prizes. This condition is that the number of players minus one is greater than one divided by the marginal cost of punishment. For the special case in which the marginal cost of reward is equal to the marginal cost of punishment, punishing the lowest performing player induces more expected effort than using any number of symmetric prizes.

Second, in asymmetric contests, we find a sufficient condition for punishment of the worst agent to induce higher expected effort than rewarding the best agent. Similar to the symmetric contests, if the number of players is large enough or if the marginal cost of punishment (to the players) is high enough, punishment induces more effort than reward. Two players are critical for the comparison of efforts: the player with the second highest ability and the player with the lowest ability. Namely, if the number of players minus two is greater or equal to the ratio of the ability of the second highest ability player to the marginal cost of punishment multiplied by the ability of the lowest ability player, then punishment will outperform reward.

In all-pay contests with perfect information, equilibria only exist in mixed strategies.[1] Siegel (2009) characterizes expected payoff of the players in all equilibria of a broad class of all-pay contests. The contests we study are all special cases of the contests covered by Siegel. We lean heavily on the payoff characterizations provided by Siegel to calculate the expected total effort of our contests.

Moldovanu et al. (2012) compare the prizes versus punishment in all-pay contests with a particular type of incomplete information. Each player's type is drawn from the same continuous distribution over all types. This assumption eliminates the payoff discontinuity, which is a primary characteristic of all-pay contests with perfect information, and consequently ensures existence of pure strategy equilibrium. The distribution of types determines the relative value of punishment and reward. More specifically, if the marginal costs of punishment and reward are equal, then it is optimal to punish the worst player if the distribution of abilities is convex.

The rest of the paper is organized as follows. In Section 2 we lay out the structure of our model and describe the two cases of rewards and punishments. In Section 3 we prove our

[1] Baye, Kovenock and de Vries (1996), Clark and Riis (1998) and González-Diaz (2012) all make important contributions to understanding equilibria of all-pay contests with perfect information.

main results: a necessary and sufficient condition for symmetric contests, and a sufficient condition for costly punishment being better than reward in asymmetric contests.

2 The model

There is a set $N = 1, ..., n$ of agents and a single principal. Agents simultaneously choose efforts $s_i \geq 0$, for $i \in N$, and then the principal makes transfers $t_i(s)$ based on $s = (s_1, ..., s_n)$ and predetermined rules $t_i(...)$. Agent i's payoff is then $u_i(s) = v + t_i(s) - s_i/a_i$, where v is a fixed utility of participation and $1/a_i$ is i's marginal cost of effort. Assume that $a_i > 0$ for all i and that $a_1 \geq \ldots \geq a_n$. The transfer rules determine a game played by the agents, which may have one or more Nash equilibria. The goal of the principal is to choose the transfer rules t_i, subject to certain constraints, to maximize the highest expected total effort of the agents in those equilibria.

We restrict attention to "winners-and-losers" mechanisms. Such a mechanism is defined by a number m of winners, where $1 \leq m \leq n-1$; a reward $r \geq 0$ given to the m winners; and a punishment $p \geq 0$ for the $n - m$ losers. The winners, given $(s_1, ..., s_n)$, are the m agents with the highest effort; ties are broken randomly and the specific randomization does not matter for the results in the paper. The remaining $n - m$ agents are losers. Each winner's transfer is $t_i(s) = r$; each loser's transfer is $t_i(s) = -p$.

The constant v does not affect the equilibrium of the game played by the agents, but could affect their decision to participate. We assume that v is large enough that agents always participate; hence v plays no further role.

Furthermore, the equilibria depend only on the gap $\Delta = r + p$ between the reward transfer and the punishment transfer and on the number m of winners. (Adding a constant k to r and subtracting this constant from p, such that the gap Δ is preserved, merely shifts payoffs up by a constant k.) Let $\widehat{V}(m, \Delta)$ be highest expected total effort of the equilibria given (m, Δ).

Of course, $\widehat{V}$ is strictly increasing in Δ, but we limit Δ as follows. A reward r implies a cost r for the principal (per recipient); a punishment p carries a cost p/γ, where $\gamma > 0$; the principal has a fixed budget b that the total cost cannot exceed. Furthermore, the principal can use only either a (pure) reward mechanism in which the punishment is 0 (hence $\Delta = r$), which we denote by (m, r); or a punishment mechanism in which the reward is 0 (hence $\Delta = p$), which we denote by $(\widetilde{m}, p)$ where $\widetilde{m} = n - m$ is the number of agents who receive the punishment. Given the budget constraint, the principal will set $r = b/m$ in a reward mechanism with m winners and $p = \gamma b/\widetilde{m}$ in a punishment mechanism with $\widetilde{m}$ losers.

Reward mechanisms thus differ by the number of winners, according to the following trade-off: as the number of winners increases, the reward r and hence the gap Δ goes down. With punishment mechanisms, the trade-off is the opposite: as the number of winners

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increases, the punish p and hence the gap Δ goes up. Fixing the number m of winners, a reward mechanism has $\Delta = r = b/m$ and a punishment mechanism has $\Delta = p = \gamma b/(n-m)$; the best mechanism is the one with the highest Δ, i.e., the punishment mechanism is better than the reward mechanism with m winners/non-losers and if and only if $\gamma b/(n-m) > b/m$, or $n/(1+\gamma) > m$.

3 Effort comparisons

In this section we establish conditions for the expected total effort of using costly punishment to be higher than using prizes. First we compare expected total effort with rewards versus punishments in a symmetric contest. Second, we address asymmetric contests and focus on the case in which all the resources are used for a single prize versus the case in which all resources are used to punish the lowest scoring player.

3.1 Symmetric contests

In symmetric contests we establish a necessary and sufficient condition for punishment inducing higher expected effort than prizes. Formally, we assume that $a_i = a$ for all i. Symmetry makes punishing a single agent much more attractive relative to rewards. In the following proposition we show that use of punishment is preferred to any number of symmetric rewards if γ is not too small. The proof of Proposition 1 is based on results established in Siegel (2009) that imply all equilibria in our symmetric contest have expected payoffs $E[u_i^*] = v$ for all $i \in N$.

Proposition 1 *In a symmetric all-pay contest, using punishment is better than reward if and only if* $\gamma > 1/(n-1)$.

Proof is done by showing two things: (1) Reward contests have the same expected effort any number of prizes, (2) Punishment contests have the highest expected effort with a single punishment. All-reward contests have the same expected effort because as m increases the size of the reward decreases (Δ), which is exactly compensated for by an increase in the probability of winning a prize. These two effects cancel each other out in all equilibria. In contrast, as the number of punishments increases, the size of the reward (Δ) decreases and the probability of punishment decreases. Both of these effects induce more expected effort as the number of punishments decreases.

Proof of Proposition 1. We begin by calculating the equilibrium expected total effort for a contest with arbitrary Δ and m prizes.

We know that in equilibrium all players' expected payoffs are such that $E[u_i^*] = v$. Take the equilibrium probabilities of winning a prize to be $\pi_i \in [0,1]$ for any player i, where $\pi_1 + ... + \pi_n = m$. Thus, $E[u_i^*] = v + \pi_i\Delta - E[s_i^*]/a$ for all i. Using the fact that in all equilibrium $E[u_i^*] = v$, $E[s_i^*] = \pi_i a$ for all i.

$$\widehat{V}(m,\Delta) = \textstyle\sum_{i=1}^{n} \pi_i a\Delta = ma\Delta. \tag{1}$$

For m rewards, $\Delta = b/m$, and (1) reduces to $\widehat{V}(m, b/m) = ab$ for any number of prizes between 1 and $n-1$. Thus, any number of rewards between 1and $n-1$ has the same total equilibrium expected effort. For the case of $n-m$ punishments, the $\Delta = \gamma b/(n-m)$, and (1) reduces to $\widehat{V}(m, \gamma b/(n-m)) = a\gamma bm/(n-m)$. Clearly, the total equilibrium expected effort with punishment is increasing in m and consequently greatest when $m = n-1$.

Therefore, we can simply compare the expected effort for the case that $m = n-1$ and $\gamma b/(n-m) > b/m$ reduces to $\gamma > 1/(n-1)$. ■

Figure 1 plots the space of parameters n and γ such that punishment induces more expected effort than rewards. It is worth noting, that for the case in which punishment is just as costly to the principal as reward ($\gamma = 1$), punishment always induces higher expected effort than reward. This is because for $\gamma = 1$, punishing one player is equivalent to rewarding $n-1$ players with the prize b. This leads to strictly more expected effort than rewarding $n-1$ players with $b/(n-1)$, which gives equivalent expected total effort as rewarding one player with b. More formally, at $\gamma = 1$, $\widehat{V}(n-1, b)$ is the expected total effort of a single punishment, $\widehat{V}(1, b)$ is the expected total effort of a single prize, and clearly $\widehat{V}(n-1, b) > \widehat{V}(n-1, b/(n-1)) = \widehat{V}(1, b)$.

3.2 Generic contests

The symmetric results do not generalize to games with asymmetric players, because $\widehat{V}$ is not necessarily monotone in the number of prizes in either reward or punishment contests. Particularly, $\widehat{V}(m, b/m)$ is not necessarily constant and $\widehat{V}(m, \gamma b/(n-m))$ is not necessarily increasing in m. This is because the number of prizes dictates which player's cost the expected total effort calculations will depend on. Consequently, the way the expected total effort changes is based on the actual player types $(a_1, ..., a_n)$. We get around this issue by narrowing the focus of this section on a comparison between a single prize contest and a single punishment contest.

Siegel (2009) introduces the concept of a generic contest and shows that in such a contest all equilibria have the same expected payoffs for all players. In our context, a contest with m prizes is *generic* if $a_m > a_{m+1}$. We assume that the players' abilities are such that $a_1 > a_2$ and $a_{n-1} > a_n$, which guarantees both the single prize and single punishment contests satisfy the

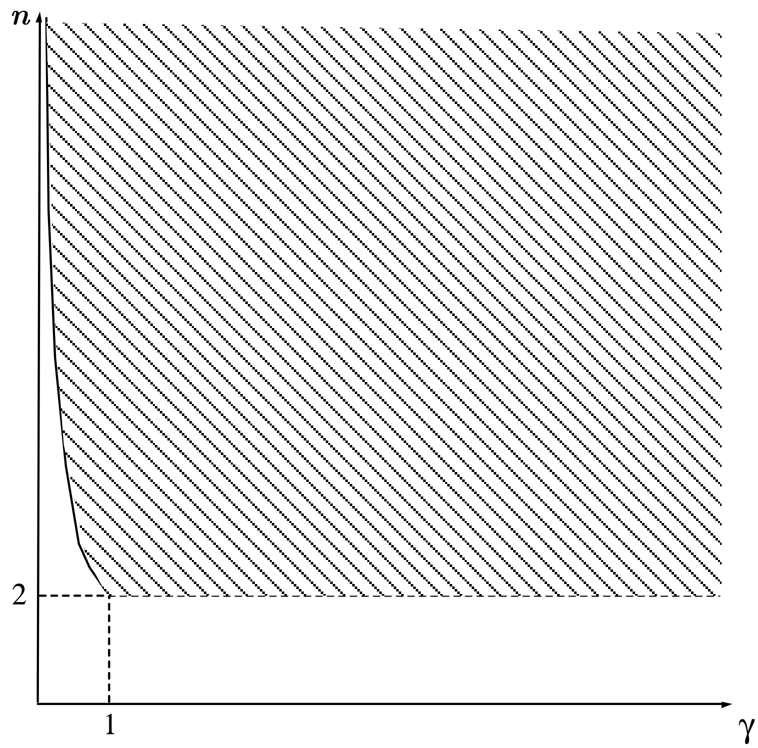

Figure 1: In the shaded region, punishing the worst agent induces more effort than any number of rewards.

generic condition. For the m prize contests the expected payoff of player i, in any equilibria, is $E[u_i^*] = v + \max\{\Delta - a_{m+1}\Delta/a_i, 0\}$.

The following proposition establishes a sufficient condition for punishment to induce more expected effort than reward.

Proposition 2 *Punishing the worst player induces more expected effort than rewarding the best, if*

$$\gamma\,(na_n - a_1) > a_2. \tag{2}$$

Proof of Proposition 2. We begin by writing out the expected total effort for the generic contest with a m prizes Δ. Denote by $(\pi_1, \pi_2, ..., \pi_n)$ the probability of each player winning the prize in some equilibria. Since each player's equilibrium expected payoff is $E[u_i^*] = v + \max\{\Delta - a_{m+1}\Delta/a_i, 0\}$, we know that $E[s_i^*] = \pi_i a_i b + a_{m+1}b - a_i b$ for all $i \in \{1, ..., m\}$ and $E[s_j^*] = \pi_j a_j b$ for all $j \in \{m+1, ..., n\}$.

$$\widehat{V}(m, \Delta) = \textstyle\sum_{i=1}^{n} \pi_i a_i \Delta + \sum_{i=1}^{m} a_{m+1}\Delta - \sum_{i=1}^{m} a_i \Delta \tag{3}$$

First we consider the total expected effort with a single reward, $\Delta = b$ and $m = 1$, which reduces (3) to

$$\widehat{V}(1, b) = \sum_{i=1}^{n} \pi_i a_i b + a_2 b - a_1 b.$$

We now establish an upper bound on the total expected effort of any equilibrium. Since $\sum_{i=1}^{n} \pi_i = 1$ and $a_1 \geq a_i$ for all i, we know that

$$\sum_{i=1}^{n} \pi_i a_i b \leq a_1 b.$$

Therefore,

$$\begin{aligned} \widehat{V}(1, b) &\leq a_1 b + a_2 b - a_1 b \\ &= a_2 b. \end{aligned}$$

Next we calculate the expected total effort for the contest with a single punishment, $\Delta = \gamma b$ and $m = n - 1$, which reduces (3) to

$$\widehat{V}(n-1, \gamma b) = \sum_{i=1}^{n} \pi_i a_i \gamma b + \sum_{i=1}^{n-1} a_n \gamma b - \sum_{i=1}^{n-1} a_i \gamma b. \tag{4}$$

Since $\sum_{i=1}^{n} \pi_i = n - 1$, the first sum in (4) is at least as big $\pi_1 = 0$ and $\pi_i = 1$ for $i = 2, ..., n$. Thus,

$$\sum_{i=1}^{n} \pi_i a_i \gamma b \geq \sum_{i=2}^{n} a_i \gamma b. \tag{5}$$

Substituting (5) into (4),

$$\begin{aligned} \widehat{V}(n-1, \gamma b) &\geq \sum_{i=2}^{n} a_i \gamma b + \sum_{i=1}^{n-1} a_n \gamma b - \sum_{i=1}^{n-1} a_i \gamma b \\ &= \gamma b \left(a_n + (n-1)a_n - a_1 \gamma\right) \\ &= \gamma b \left(n a_n - a_1\right). \end{aligned}$$

Based on the inequalities above, punishing a single player yields more expected effort than rewarding a single player if $\gamma \left(n a_n - a_1\right) > a_2$. ∎

The condition used in the proposition provides a straightforward interpretation of how the number of players and distribution of players' abilities impact the value of punishment versus reward. We set $\gamma = 1$ (punishment is just as costly to the principal as reward) and for an arbitrary number of players n plot the space of abilities such that punishment is better than reward.

We manipulate (2) to find a sufficient condition more similar to what we found in the symmetric case

$$n > \left(\frac{a_2}{a_n}\right) \frac{1}{\gamma} + \frac{a_1}{a_n}. \tag{6}$$

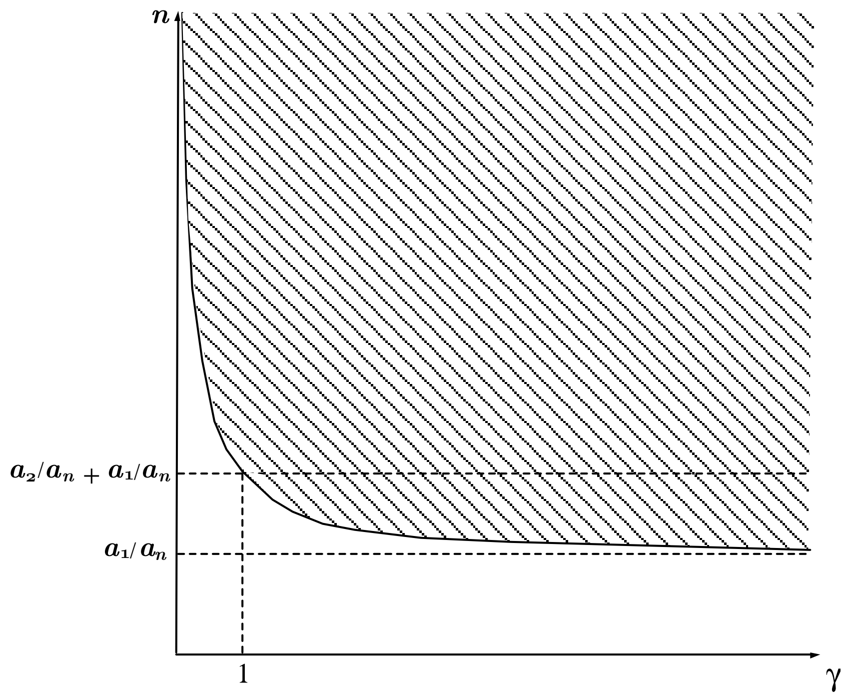

Figure 2: In the shaded region, punishing the worst agent induces more effort than any number of rewards.

In Figure 2, we plot condition (6) in $n \times \gamma$ space. Similar to the symmetric case, if the set of players is sufficiently large or the marginal effectiveness of punishment is high enough, then punishment induces more expected effort that reward.

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