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Constant population growth rate and time to build: the Solow model case

Luca Guerrini
University of Bologna

Abstract

We analyze the Solow model with constant population growth rate, where production occurs with a time delay while new capital is installed (time to build). Taking time delay as a bifurcation parameter, we discuss the stability and the existence of the Hopf bifurcation occurring at the equilibrium. It is proved that the equilibrium is locally asymptotically stable for any delay if population is increasing, whereas there are stability switches if population is decreasing. In this case, a steady state bifurcation and a Hopf bifurcation occur. Contrary to the Solow model, a different scenario arises.

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Contact: Luca Guerrini - luca.guerrini@unibo.it.
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1. Introduction

Ritschl (1985) discussed the problem of existence and stability of the steady state in the Solow (1956) model. He found that a steady state exists if the rate of population change and the savings ratio have the same sign. The occurrence of an unstable equilibrium when population is decreasing suggested Ritschl (1985) to modify the shape of the saving function employed in order to guarantee existence and stability of the steady state with any population growth rate. Recently, a negative population growth rate hypothesis was addressed for the Rebelo (1991) AK growth model by Ferrara (2011), and for a model of semi-endogenous growth by Christiaans (2011).

In this paper, we propose to generalize Ritschl (1985) model, and so the Solow (1956) model, by assuming that production occurs with a delay while new capital is installed. In the literature this assumption is often referred as "time to build". The idea of introducing a time delay into dynamics of economic processes was first posed rigorously in Kalecki (1935), and later revived by Kydland and Prescott (1982). More recently, Asea and Zak (1999), Zak (1999), Szydłowski (2003), and Krawiec and Szydłowski (2004) analyzed the time to build assumption in the Solow (1956) growth model, and proved the steady state to exhibit Hopf cycles. Differently from these existing contributions in the literature, we depart from their assumption of absence of population growth and consider a positive or negative population growth rate hypothesis. From a mathematical point of view, the resulting model is governed by a delay differential equation with delay dependent coefficients, instead of being ruled by a delay differential equation with constant coefficients.

Taking the delay as a bifurcation parameter, the stability and Hopf bifurcation of the model are discussed by analyzing the corresponding transcendental characteristic equation of its linearized equation. It is well known that a sufficient condition for the local asymptotic stability of the equilibrium is that each of the characteristic roots has negative real parts. Thus, the marginal stability is determined by the zero eigenvalue and the purely imaginary eigenvalues. Our analysis is conducted using the Poincaré-Andronov-Hopf bifurcation theorem, which has been generalized for delay differential equations (see, e.g., Hale and Verduyn Lunel, 1993). This theorem predicts a bifurcation to a limit cycle when the characteristic equation has a pair of simple complex conjugate eigenvalues at a critical value of the delay parameter and the eigenvalues cross the imaginary axis with non-zero speed (transversality condition), i.e. the derivative of the real part of eigenvalues with respect to the delay parameter is non-zero.

When the population growth rate is positive, it is found that the model predicts the same results as the model with a stationary population (zero population growth), i.e. the equilibrium is locally asymptotically stable. On the other hand, if the population growth rate is negative, the equilibrium may change its stability and two different types of bifurcations take place: a steady state bifurcation and a Hopf bifurcation. As it can be inferred from these results, the introduction of a time lagged accumulation of capital in the Solow growth model with nonconstant population gives rise to a wide variety of dynamic behaviors.

2. The model

We present the time to build version of the Ritschl (1985) model. Specifically, we consider the Solow (1956) model with constant (positive or negative) population growth where the Cobb-Douglas technology displays a delay of τ periods before capital can be used for production. The new capital available for use at time t is produced using capital available at time $t-\tau$. If we also assume the absence of capital depreciation, then the change in capital stock is given by $\dot{K} = sK_d^{\alpha}L^{1-\alpha}$, where K denotes physical capital, $K_d = K(t-\tau)$, L represents population, $\alpha \in (0,1)$

is capital's share, and s is the constant saving rate. Setting $k = K/L$, the capita accumulation equation becomes

$$\dot{k} = s\left(L^{-1}L_d\right)^{\alpha} k_d^{\alpha} - \frac{\dot{L}}{L}\,k. \tag{1}$$

Normalizing the number of people at time zero to one, we find that $L = e^{nt}$, where $n = \dot{L}/L \neq 0$ denotes the constant population growth rate. Plugging L into Eq. (1), we obtain the following delay differential equation with delay dependent coefficients:

$$\dot{k} = se^{-\alpha n\tau}k_d^{\alpha} - nk. \tag{2}$$

Equilibria, or steady states in the language of the economical sciences, of Eq. (2) coincide with those for $\tau = 0$. It is found that there exists a unique positive steady state k_* for which $sk_*^{\alpha-1} = n$, when the rate of population change and the savings ratio have the same sign. To investigate the local stability of Eq. (2), we linearise it in the vicinity of the equilibrium point k_* and get

$$\dot{k} = -n\left(k - k_*\right) + \alpha ne^{-\alpha n\tau}\left(k_d - k_*\right). \tag{3}$$

The associated characteristic equation, obtained by substituting $k - k_* = e^{\lambda t}$ into Eq. (3), is of the form

$$P(\lambda) = \lambda + n - \alpha ne^{-\alpha n\tau}e^{-\lambda\tau} = 0. \tag{4}$$

Eq. (4) is a transcendental equation having in general an infinite number of complex roots. As well, the eigenvalues λ defined from the characteristic equation depend in a continuous fashion on the delay time τ.

3. Stability analysis and existence of Hopf bifurcation

In this section, we investigate stability and existence of Hopf bifurcation for Eq. (2), assuming τ as a bifurcation parameter. We start by considering, as usual, the case without delay. When $\tau = 0$, the characteristic equation (4) turns out to be $\lambda = -(1-\alpha)n$. Therefore, Eq. (2) is locally asymptotically stable for $n > 0$, and unstable for $n < 0$. Next, let us consider the case $\tau > 0$. It is well-known that the equilibrium of Eq. (2) is locally asymptotically stable if its characteristic roots have negative real parts. Hence, the marginal stability is determined by the values $\lambda = 0$ and $\lambda = i\omega$ $(\omega > 0)$.

3.1. The case $\boldsymbol{\lambda}$= 0

Lemma 1.

i) If $n > 0$, $\lambda = 0$ is not a root of Eq. (4).

ii) If $n < 0$, $\lambda = 0$ is a simple root of Eq. (4) for some $\tau \neq \bar{\tau} = -1/n$ such that $\alpha e^{-\alpha n\tau} = 1$.

Proof. For $\lambda = 0$, Eq. (4) becomes $\alpha e^{-\alpha n\tau} = 1$. The first part of the statement follows from being $\alpha \in (0,1)$ and $e^{-\alpha n\tau} > 1$ if $n < 0$ and $e^{-\alpha n\tau} < 1$ if $n > 0$. It remains to show that $\lambda = 0$ is a simple root when $n < 0$ and $\tau \neq \bar{\tau} = -1/n$. If $\lambda = 0$ is a repeated root, then $P(0) = P'(0) = 0$ holds true. This leads to the contradiction $1 + n\tau = 0$, concluding the proof. □

Remark 1. When $\tau = \bar{\tau}$, $\lambda = 0$ is a repeated root for Eq. (4). In this case, our system becomes a degenerated case where is very difficult to determine the crossing direction of the characteristic roots through the imaginary axis.

Let $n < 0$. Let τ_c denote $\tau \neq \bar{\tau} = -1/n$ for which $\alpha e^{-\alpha n\tau} = 1$. According to Lemma 1, the hypothesis for a Hopf bifurcation to occur at τ_c are verified except for the transversality condition.

Differentiating Eq. (4) with respect to τ we obtain

$$\frac{d\lambda}{d\tau} = -\frac{(\lambda + n)(\lambda + \alpha n)}{1 + (\lambda + n)\tau}, \tag{5}$$

which implies

$$\left.\frac{d\lambda}{d\tau}\right|_{\tau=\tau_c} = -\frac{\alpha n^2}{1 + n\tau_c} = \frac{\alpha n^2 \bar{\tau}}{\tau_c - \bar{\tau}} \neq 0.$$

Hence, the transversality condition is fulfilled. The sign of $(d\lambda/d\tau)_{\tau=\tau_c}$ is positive if $\tau_c > \bar{\tau}$, and it is negative if $\tau_c < \bar{\tau}$. Furthermore, when τ crosses τ_c, the number of roots with positive real parts of the characteristic equation (4) increases if $\tau_c > \bar{\tau}$, and it decreases if $\tau_c < \bar{\tau}$.

3.2. The case $\lambda = i\omega$

Let $\lambda = i\omega$ $(\omega > 0)$ be a root of Eq. (4). Substituting it into (4), and separating the real and imaginary parts gives

$$\omega = -\alpha n e^{-\alpha n \tau} \sin \omega\tau, \quad n = \alpha n e^{-\alpha n \tau} \cos \omega\tau. \tag{6}$$

By squaring and adding these equations, we find ω to be solution of

$$\omega^2 = \left(\alpha n e^{-\alpha n \tau}\right)^2 - n^2. \tag{7}$$

We remark that if Eq. (7) has no positive roots, then Eq. (2) is delay independent stable or unstable for any given time delay depending on whether the system which is free of time delay is stable or not. In contrast, if Eq. (7) has a positive root ω, then

$$\omega = \sqrt{\left(\alpha n e^{-\alpha n \tau}\right)^2 - n^2}, \tag{8}$$

and we must have $\left|\alpha n e^{-\alpha n \tau}\right| > |n|$.

Lemma 2. Eq. (7) has no positive real root when $n > 0$, whereas it has positive roots if $n < 0$ and $\tau > \tau_c$, where $\tau_c > 0$ is such that $\alpha e^{-\alpha n \tau_c} = 1$.

We need now to determine the changing direction of the real part of characteristic roots as the parameter τ varies. Let $n < 0$ and $\tau > \tau_c$. Substituting (8) into Eqs. (6) yields

$$\begin{cases} \sqrt{\left(\alpha n e^{-\alpha n \tau}\right)^2 - n^2} & = & -\alpha n e^{-\alpha n \tau} \sin\left[\tau \sqrt{\left(\alpha n e^{-\alpha n \tau}\right)^2 - n^2}\right], \\ n & = & \alpha n e^{-\alpha n \tau} \cos\left[\tau \sqrt{\left(\alpha n e^{-\alpha n \tau}\right)^2 - n^2}\right]. \end{cases} \tag{9}$$

If system (9) has simple positive roots $\tau_j(\tau)$ in the interval (τ_c, ∞) for some $j = 0, 1, 2, ..$, then a pair of simple conjugate pure imaginary roots $\lambda = \pm i\omega(\tau)$ of Eq. (4) exists at $\tau = \tau_j(\tau)$. The occurrence of stability switches, i.e. crossing of the imaginary axis, takes place at the zeros of the functions

$$S_j(\tau) = \tau - \tau_j(\tau), \text{ for some } j = 0, 1, 2, ... \tag{10}$$

Proposition 1. Let $n < 0$ and $\tau > \tau_c$. The characteristic equation (4) admits a pair of simple conjugate pure imaginary roots $\lambda = \pm i\omega(\tau_j)$, $\omega(\tau_j) > 0$, at $\tau_j \in (\tau_c, \infty)$ if $S_j(\tau) = 0$, for some $j = 0, 1, 2, ...$ The crossing direction through the imaginary axis is determined by

$$sign\left[\left.\frac{d\,(Re\lambda)}{d\tau}\right|_{\lambda=i\omega(\tau_j)}\right] = sign\left[\left.\frac{dS_j(\tau)}{d\tau}\right|_{\tau=\tau_j}\right]. \tag{11}$$

Proof. The existence of pure imaginary roots follows from Lemma 2. Let $\lambda = i\omega(\tau)$ $(\omega > 0)$ be a root of Eq. (4). This root is simple because $P'(i\omega) = 1 + n\tau + i\tau\omega \neq 0$. From Eq. (5), we know

$$\left.\frac{d\lambda}{d\tau}\right|_{\lambda=i\omega(\tau_j)} = Re\left[-\frac{(i\omega+n)(i\omega+\alpha n)}{[1+(i\omega+n)\tau]}\right].$$

Hence,

$$Re\left(\frac{d\lambda}{d\tau}\right)_{\lambda=i\omega(\tau_j)} = \frac{\omega^2 - \alpha n\left[n + \left(\alpha n e^{-\alpha n\tau}\right)^2\tau\right]}{(1+n\tau)^2 + (\omega\tau)^2}. \tag{12}$$

From Eqs. (6), we have

$$\tau_j = \frac{\tan^{-1}\left[-\frac{\omega(\tau)}{n}\right] + 2j\pi}{\omega(\tau)}, \quad j = 0, 1, 2, ...$$

Then,

$$\frac{d\tau_j}{d\tau} = -\frac{\left[n + \left(\alpha n e^{-\alpha n\tau}\right)^2\tau_j\right]\omega\omega'}{\left(\alpha n e^{-\alpha n\tau}\right)^2\omega^2}.$$

Differentiating Eq. (10) with respect to τ, and using $\omega\omega' = -\alpha n\left(\alpha n e^{-\alpha n\tau}\right)^2$, we get

$$\frac{dS_j(\tau)}{d\tau} = 1 - \frac{d\tau_j}{d\tau} = \frac{\omega^2 - \alpha n\left[n + \left(\alpha n e^{-\alpha n\tau}\right)^2\tau_j\right]}{\omega^2}.$$

Comparing this result with Eq. (12) yields the statement. □

The stability of Eq. (2) can now be determined by the roots of Eq. (10) and by the sign of $[dS_j(\tau)/d\tau]_{\tau=\tau_j}$. If the sign is positive, each crossing of the real part of characteristic roots at τ_j must be from left to right. If the sign is negative, the real part of a pair of conjugate roots of Eq. (4) changes from a positive value to a negative value when τ_j is crossed. Hence, the characteristic equation (4) has a pair of simple complex conjugate pure imaginary roots at $\tau \in \Omega = \{\tau_0, \tau_1, \tau_2, ..., \tau_N\}$, where $N \in \mathbb{N}$. If $\tau \in \Omega$ is not a zero point for $dS_j(\tau)d\tau$, then Eq. (11) yields $d\left(Re\lambda\right)/d\tau \neq 0$. In conclusion, all the hypothesis for a Hopf bifurcation to emerge at $\tau \in \Omega$ are satisfied.

We can proceed to state the main result of this paper.

Theorem 1.

1. Let $n > 0$. The positive equilibrium k_* of Eq. (2) is locally asymptotically stable for all $\tau \geq 0$.

2. Let $n < 0$. Let τ_c be such that $\alpha e^{-\alpha n\tau_c} = 1$, $\tau_c \neq -1/n$, and let S_j be defined as in (10).

   i) The positive equilibrium k_* of Eq. (2) is unstable for all $\tau \in [0, \tau_c]$, whereas it may change its stability finitely many times for $\tau \in (\tau_c, \infty)$ depending on the sign of $dS_j(\tau)/d\tau$.

   ii) The positive equilibrium k_* of Eq. (2) undergoes a steady state bifurcation at $\tau = \tau_c$, and a Hopf bifurcation at $\tau \in \Omega$ if τ is not a zero point of $dS_j(\tau)/d\tau$.

4. Conclusion

This paper analyzes the neoclassical growth model with constant population growth when the time to build assumption is introduced through a delay differential equation for capital. It has been proved that the system may display stability switches and generate Hopf bifurcations. Compared to the case of zero population growth, the dynamics are richer and depend on the population growth rate. Since many industrialized countries experience a negative rate of population growth, the present work aims to expand economic growth theory in this area, focusing on the time to build structure and size of a population and their influence on economic growth. A declining population size has the effect to destabilise the equilibrium of the Solow's model and to generate oscillations via Hopf bifurcations. The fact to know that a Hopf bifurcation exists nothing says about the stability properties of the involved periodic orbits. For future research it would be interesting to analyze whether the bifurcation is supercritical or subcritical, i.e. whether the periodic orbit is locally stable or unstable.

References

Asea, P. and P. Zak (1999) "Time to Build and Cycles" *Journal of Economic Dynamic and Control* **23**, 1155-1175.

Christiaans, T. (2011) "Semi-Endogenous Growth when Population is Decreasing" *Economics Bulletin* **31**, 2667-2673.

Ferrara, M. (2011) "An AK Solow-Swan Model with a Non-Positive Rate of Population Growth" *Applied Mathematical Sciences* **5**, 1241-1244.

Hale, J.K. and S.M. Verduyn Lunel (1993) *Introduction to Functional Differential Equations*, Springer Verlag, Berlin, New York.

Kalecki, M. (1935) "A Macroeconomic Theory of Business Cycles" *Econometrica* **3**, 327-344.

Kydland, F.E. and E.C. Prescott (1982) "Time to Build and Aggregate Fluctuations" *Econometrica* **50**, 1345-1370.

Krawiec, A. and M. Szydłowski (2004) "A Note on the Kaleckian Lags in the Solow Model" *Review of Political Economy* **16**, 501-506.

Rebelo, S. (1991) "Long-Run Policy Analysis and Long-Run Growth" *Journal of Political Economy* **99**, 500-521.

Ritschl, A. (1985) "On the Stability of the Steady-State when Population is Decreasing" *Journal of Economics* **45**, 161-170.

Solow, R.M. (1956) "A Contribution to the Theory of Economic Growth" *Quarterly Journal of Economics* **70**, 65-94.

Szydłowski, M. (2003) "Time-to-Build in Dynamics of Economic Models II: Models of Economic Growth" *Chaos, Solitons & Fractals* **14**, 355-364.

Zak, P.J. (1999) "Kaleckian Lags in General Equilibrium" *Review of Political Economy* **11**, 321-330.