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A Note on the Moments of the Skew-Normal Distribution

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Abstract

Azzalini's skew-normal distribution is an attractive tool for modeling the skewness observed in many economic and financial variables. Formulas for the odd moments of the skew-normal distribution have been given by Henze (1986) and, more recently, Martinez et al. (2008). This note provides a rather straightforward alternative approach to the calculation of the odd moments of the skew-normal distribution. It exploits a striking similarity between the density and the moment generating function of a skew-normal variable and leads to an attractive expression for the odd moments.

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1. Introduction

Random variable Z has a standard skew–normal (SN) distribution if its density is given by

$$f_Z(z;\gamma) = 2\phi(z)\Phi(\gamma z), \quad \gamma \in \mathbb{R}, \tag{1}$$

where

$$\phi(z) = \frac{e^{-z^2/2}}{\sqrt{2\pi}} \quad \text{and} \quad \Phi(z) = \int_{-\infty}^{z} \phi(\xi)d\xi$$

are the standard normal pdf and cdf, respectively. Density (1) appears in O'Hagan and Leonard (1976) and was independently proposed and systematically investigated by Azzalini (1985, 1986). The density (1) can be viewed as a normal pdf times a weight factor $2\Phi(\gamma z)$ which depends on the *asymmetry parameter* γ. If $\gamma < 0$, the weight will be larger for negative z, leading to a negatively skewed density, whereas $\gamma > 0$ results in positive skewness. As put forward by Azzalini (1985), an interesting feature of (1), beyond its mathematical tractability, is its "strict inclusion" of the Gaussian for $\gamma = 0$, i.e., the Gaussian does not arise as a limit case. These features make the SN a popular candidate for capturing the asymmetries observed in the distribution of many economic and financial variables (e.g., Adcock and Shutes, 2005; Christodoulakis and Peel, 2009; and Harvey *et al.*, 2010). There is also a growing interest in the application of finite mixtures of SN distributions (e.g., Lin *et al.*, 2007; Lin, 2009; Frühwirth–Schnatter and Pyne, 2010; Bernardi, 2012; Augustyniak and Boudreault, 2012; and Haas, 2010).

Application of the SN distribution in finance is also facilitated by the availability of convenient routines for computing its cdf, as required, for example, for the calculation of Value–at–Risk (VaR). The cdf of the SN distribution (1) can be written as

$$F_Z(z;\gamma) = 2\int_{-\infty}^{z} \phi(\xi)\Phi(\gamma\xi)d\xi = 2\int_{-\infty}^{z}\int_{-\infty}^{\gamma\xi} \phi(\xi)\phi(\zeta)d\zeta d\xi,$$

which shows the close link between the cdf of the SN and that of the bivariate normal distribution, which has been extensively studied. A recent discussion is provided by Castellares *et al.* (2012). These authors also derive a power series expansion for the quantile function of the SN distribution, which can directly be used to compute the VaR.

Moments are frequently used to characterize the properties of a specific distribution, such as the mean, variance, skewness, and kurtosis. Azzalini (1985) has shown that the even moments of the SN are equal to those of the standard normal, and he also calculated the first two odd moments. General formulas for the odd moments were provided by Henze (1986) and Martínez *et al.* (2008). In the next section, an alternative derivation based on the moment generating function is provided which gives rise to an attractive expression for the odd moments of (1); the relation between the different expressions is also addressed.

2. Odd moments of the SN distribution

The computation of the odd moments of the SN distribution can be based on the moment generating function (mgf). Using a result of Ellison (1964), Azzalini (1985) showed that the mgf of the SN distribution is

$$m(t) = 2e^{t^2/2}\Phi(\delta t) = 2\psi(t)\Phi(\delta t), \quad \delta = \frac{\gamma}{\sqrt{1+\gamma^2}}, \tag{2}$$

where $\psi(t) = e^{t^2/2}$ is the mgf of the standard normal distribution. In view of the invariance of the Gaussian function with respect to Fourier transformation there is an intriguing similarity between (1) and (2) which can directly be exploited for the evaluation of the moments. Let $\ell \in \mathbb{N}_0$. Using Leibniz' rule for the differentiation of a product,

$$\begin{aligned} m^{(2\ell+1)}(t) &= 2\sum_{i=0}^{2\ell+1}\binom{2\ell+1}{i}\Phi^{(i)}(\delta t)\psi^{(2\ell-1-i)}(t) \\ &= \Phi(\delta t)\psi^{(2\ell-1)}(t) + \delta\sqrt{\frac{2}{\pi}}\sum_{i=1}^{2\ell+1}\binom{2\ell+1}{i}\varphi^{(i-1)}(t;\delta^2)\psi^{(2\ell-1-i)}(t), \end{aligned}$$

where the last equation uses the fact that

$$\Phi^{(i)}(\delta t) = \frac{\delta}{\sqrt{2\pi}}\frac{d^{i-1}}{dt^{i-1}}\left(e^{-t^2\delta^2/2}\right) = \frac{\delta}{\sqrt{2\pi}}\varphi^{(i-1)}(t;\delta^2), \quad i \geq 1,$$

where $\varphi(t;\delta^2) = e^{-t^2\delta^2/2}$ is the characteristic function (cf) of the normal distribution with mean zero and variance δ^2. Hence, since the odd moments of normals with zero mean are zero,

$$\begin{aligned} \mathrm{E}(Z^{2\ell+1}) &= m^{(2\ell+1)}(0) = \delta\sqrt{\frac{2}{\pi}}\sum_{i=0}^{\ell}\binom{2\ell+1}{2i+1}\varphi^{(2i)}(0;\delta^2)\psi^{(2(\ell-i))}(0) \\ &= \delta\sqrt{\frac{2}{\pi}}\sum_{i=0}^{\ell}\frac{(2\ell+1)!}{(2i+1)!(2(\ell-i))!}(\sqrt{-1})^{2i}\delta^{2i}\frac{(2i)!}{i!2^i}\frac{(2(\ell-i))!}{(\ell-i)!2^{\ell-i}} \\ &= \sqrt{\frac{2}{\pi}}\frac{(2\ell+1)!}{2^\ell \ell!}\sum_{i=0}^{\ell}(-1)^i\binom{\ell}{i}\frac{\delta^{2i+1}}{2i+1}. \end{aligned} \tag{3}$$

In applications, interest often centers on the first four moments (as required for skewness and kurtosis); this involves the first two odd moments,

$$\mathrm{E}(Z) = \sqrt{\frac{2}{\pi}}\delta, \quad \mathrm{E}(Z^3) = \sqrt{\frac{2}{\pi}}(3\delta - \delta^3). \tag{4}$$

From (4), central moments $\kappa_i := \mathrm{E}[(Z - \mathrm{E}(Z))^i]$ are

$$\begin{aligned} \kappa_2 &= \mathrm{Var}(Z) = 1 - \frac{2}{\pi}\delta^2, \quad \kappa_3 = \sqrt{\frac{2}{\pi}}\frac{4-\pi}{\pi}\delta^3, \\ \kappa_4 &= 3 - \frac{12}{\pi}\delta^2 + \frac{8\pi - 12}{\pi^2}\delta^4 = 3\left(1 - \frac{2}{\pi}\delta^2\right)^2 + \frac{8\pi-24}{\pi^2}\delta^4, \end{aligned}$$

and thus the coefficients of skewness and kurtosis are (cf. Azzalini, 1985)

$$\text{skew.} = \frac{\kappa_3}{\kappa_2^{3/2}} = \frac{\sqrt{2}(4-\pi)\delta^3}{(\pi - 2\delta^2)^{3/2}}, \quad \text{and} \quad \text{kurt.} = \frac{\kappa_4}{\kappa_2^2} = 3 + \frac{8(\pi-3)\delta^4}{(\pi-2\delta^2)^2},$$

respectively. Note that skewness is restricted to the interval $(-0.995, 0.995)$, whereas kurtosis is between 3 and its maximum value 3.869 which is approached as $\delta \to \pm 1$ (i.e., $\gamma \to \pm\infty$) (Azzalini, 1985).

For further illustration, the fifth and seventh moments in the form (3) turn out to be

$$\begin{aligned} \mathrm{E}(Z^5) &= \sqrt{\frac{2}{\pi}}(15\delta - 10\delta^3 + 3\delta^5) \\ \mathrm{E}(Z^7) &= \sqrt{\frac{2}{\pi}}(105\delta - 105\delta^3 + 63\delta^5 - 15\delta^7). \end{aligned}$$

One may finally note that expression (3) for the odd moments is also obtained (though much more tediously) by directly expanding the mgf, i.e.,

$$\begin{aligned} m(t) &= 2e^{t^2/2}\Phi(\delta t) = e^{t^2/2} + \sqrt{\frac{2}{\pi}} e^{t^2/2} \sum_{i=0}^{\infty} (-1)^i \frac{\delta^{2i+1} t^{2i+1}}{2^i i!(2i+1)} \\ &= e^{t^2/2} + \sqrt{\frac{2}{\pi}} \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \frac{(-1)^i \delta^{2i+1} t^{2(i+j)+1}}{2^{i+j} i! j! (2i+1)} \\ &\overset{\ell=i+j}{=} e^{t^2/2} + \sqrt{\frac{2}{\pi}} \sum_{\ell=0}^{\infty} t^{2\ell+1} \left\{ \sum_{i=0}^{\ell} \frac{(-1)^i \delta^{2i+1}}{2^\ell i!(\ell-i)!(2i+1)} \right\}. \end{aligned}$$

In applications, one will typically deal with the variable $Y = \mu + \sigma Z$, with μ and σ being parameters of location and scale, respectively. The density of Y is

$$f_Y(y; \mu, \sigma, \gamma) = \frac{2}{\sigma}\phi\left(\frac{x-\mu}{\sigma}\right)\Phi\left(\gamma\frac{x-\mu}{\sigma}\right),$$

and its moments are given by

$$M_n(\mu, \sigma, \gamma) := \mathrm{E}(Y^n) = \mathrm{E}\{(\mu + \sigma Z)^n\} = \sum_{i=0}^{n} \binom{n}{i} \mathrm{E}(Z^i)\mu^{n-i}\sigma^i.$$

3. Comparison with earlier expressions

The first general expression for the odd moments of the SN distribution was provided by Henze (1986), who showed that an SN random variable Z has representation

$$Z = \delta|U| + \sqrt{1-\delta^2}V, \tag{5}$$

where U and V are independent standard normal variables. Then the binomial formula leads to

$$\mathrm{E}(Z^{2\ell+1}) = \sqrt{\frac{2}{\pi}} \frac{(2\ell+1)!}{2^\ell} \frac{\gamma}{(1+\gamma^2)^{\ell+1/2}} \sum_{m=0}^{\ell} \frac{m!(2\gamma)^{2m}}{(2m+1)!(\ell-m)!}. \tag{6}$$

To directly see that (3) and (6) are identical, substitute $\delta = \gamma/\sqrt{1+\gamma^2}$ to write the sum in (3) in terms of γ, i.e.,

$$\begin{aligned}\sum_{i=0}^{\ell} \binom{\ell}{i} \frac{(-1)^i \delta^{2i+1}}{(2i+1)} &= \frac{\gamma}{(1+\gamma^2)^{\ell+1/2}} \sum_{i=0}^{\ell} \binom{\ell}{i} \frac{(-1)^i \gamma^{2i}(1+\gamma^2)^{\ell-i}}{(2i+1)} \\ &= \frac{\gamma}{(1+\gamma^2)^{\ell+1/2}} \sum_{i=0}^{\ell} \sum_{j=0}^{\ell-i} \binom{\ell}{i} \binom{\ell-i}{j} \frac{(-1)^i \gamma^{2(i+j)}}{(2i+1)} \\ &\overset{m=i+j}{=} \frac{\ell!\gamma}{(1+\gamma^2)^{\ell+1/2}} \sum_{m=0}^{\ell} \left\{ \frac{1}{(\ell-m)!} \sum_{i=0}^{m} \frac{(-1)^i}{i!(m-i)!(2i+1)} \right\} \gamma^{2m}.\end{aligned}$$

Comparing coefficients of γ^{2m}, equality of (3) and (6) follows from the combinatoric identity

$$\sum_{i=0}^{m} \binom{m}{i} \frac{(-1)^i}{2i+1} = \frac{(2^m m!)^2}{(2m+1)!} = \frac{2^{2m}}{\binom{2m}{m}(2m+1)},$$

which is well–known (e.g., Paolella, 2006, p. 20).

A further method for calculating the odd moments has rather recently been proposed by Martínez *et al.* (2008), who observed that integration by parts leads to the recursive formula

$$\mathrm{E}(Z^{2\ell+1}) = 2\int_{-\infty}^{\infty} z^{2\ell+1}\phi(z)\Phi(\gamma z)dz = 2\ell\mathrm{E}(Z^{2\ell-1}) + \sqrt{\frac{2}{\pi}} \frac{\gamma}{(1+\gamma^2)^{\ell+1/2}} \frac{(2\ell)!}{2^\ell \ell!}. \quad (7)$$

Relation (7) also appears in Equation (2.4) in Pal *et al.* (2008), and Chiogna (1998) used a similar recursion to calculate the incomplete moments of the SN distribution.

Solving recursion (7) gives

$$\begin{aligned}\mathrm{E}(Z^{2\ell+1}) &= \sqrt{\frac{2}{\pi}} \sum_{i=0}^{\ell} \frac{2^\ell \ell!}{2^i i!} \frac{(2i)!}{2^i i!} \frac{\gamma}{(1+\gamma^2)^{i+1/2}} \\ &= \sqrt{\frac{2}{\pi}} \frac{\gamma}{(1+\gamma^2)^{\ell+1/2}} 2^\ell \ell! \sum_{m=0}^{\ell} \left\{ \sum_{i=0}^{\ell-m} \binom{\ell-i}{m} \binom{2i}{i} \frac{1}{2^{2i}} \right\} \gamma^{2m}, \quad (8)\end{aligned}$$

which, by comparison with (6), gives rise to the combinatorial identity, for $\ell \geq m$,

$$E_{\ell,m} := \sum_{i=0}^{\ell-m} \binom{\ell-i}{m} \binom{2i}{i} \frac{1}{2^{2i}} = \frac{(2\ell+1)!m!}{2^{2(\ell-m)}\ell!(2m+1)!(\ell-m)!} = \frac{1}{2^{2(\ell-m)}} \frac{\binom{2\ell+1}{\ell}\binom{\ell+1}{m+1}}{\binom{2m+1}{m}}, \quad (9)$$

which can be directly verified by straightforward calculation: Identity (9) is clearly true for $\ell = m$; for $m = 0$, (9) becomes

$$E_{\ell,0} = \sum_{i=0}^{\ell} \binom{2i}{i} \frac{1}{2^{2i}} = \frac{2\ell+1}{2^{2\ell}} \binom{2\ell}{\ell}, \quad (10)$$

which is Identity (1.109) in Gould (1972).[1] Along with these observations, the identity (9) then follows from the fact that both sides of (9) satisfy the recursion

$$E_{\ell,m} = E_{\ell-1,m-1} + E_{\ell-1,m}, \quad \ell > m.$$

E.g.,

$$\begin{aligned} E_{\ell,m} &= \sum_{i=0}^{\ell-m} \binom{\ell-i}{m}\binom{2i}{i}\frac{1}{2^{2i}} = \sum_{i=0}^{\ell-m}\left\{\binom{\ell-1-i}{m-1}+\binom{\ell-1-i}{m}\right\}\binom{2i}{i}\frac{1}{2^{2i}} \\ &= \underbrace{\sum_{i=0}^{(\ell-1)-(m-1)}\binom{\ell-1-i}{m-1}\binom{2i}{i}\frac{1}{2^{2i}}}_{=E_{\ell-1,m-1}} + \underbrace{\sum_{i=0}^{\ell-1-m}\binom{\ell-1-i}{m}\binom{2i}{i}\frac{1}{2^{2i}}}_{=E_{\ell-1,m}}. \end{aligned}$$

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[1] Identity (10) can also be used for calculating the expectation in Banach's matchbox problem (cf. Feller, 1957, p. 212; Paolella, 2006, p. 162).

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