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# The Shapley value for fuzzy games: TU games approach

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## Abstract

In this note we investigate the Shapley value for fuzzy games proposed by Hwang and Liao (2009). We show that there exists a transferable-utility (TU) decomposition games that can be adopted to characterize the fuzzy Shapley value, i.e., the fuzzy Shapley value consists of the Shapley value of the corresponding TU decomposition games.

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# 1 Introduction

The *Shapley value* (1953) provides an efficient and fair cost allocation mechanism for sharing costs between products (or factors) on standard TU games. In the framework of *fuzzy games*, there are several extensions of the Shapley value in the literature. Butnariu (1980) defined a Shapley function as a function which maps a fuzzy game to a function deriving the Shapley value from a fuzzy coalition. He showed the explicit form of the Shapley function on a limited class of fuzzy games. Later, Tsurumi et al. (2001) followed Butnariu's approach. They introduced and investigated a more natural class of fuzzy games. Axioms of the Shapley function are renewed and an explicit form of the Shapley function on the natural class is given. Branzei et al. (2003) defined the Shapley value as the average of all marginal vectors which is generated by the orders of all players. According to consistency which related to the self-reduced game due to Hart and Mas-Colell (1989), Hwang and Liao (2009) provided a definition of the consistent value which is a generalization of the Shapley value of fuzzy games. Here, we focus on the fuzzy Shapley value due to Hwang and Liao (2009), which we name as the *consistent value*.

In a standard TU game, each agent is either fully involved or not involved at all in participation with some other agents, while in a fuzzy TU game, each player is allowed to participate with infinite many different activity levels. Hence, fuzzy TU games constitute a generalization of standard TU games. These mentioned above raise one question in the framework of fuzzy TU games:

- whether the consistent value on fuzzy TU games could be described in the framework of standard TU games.

The note is aimed at answering the question. We firstly introduce the notion of *TU decomposition games* for a fuzzy game and adopt it to study the consistent value. We further show that the consistent value consists of the Shapley value of the corresponding TU decomposition games.

# 2 Preliminaries

Let $U$ be the universe of players. For $i \in U$ and $f_i \in [0,1]$, we set $F_i = [0, f_i]$ as the action space of player $i$, where the action 0 denotes no participation, and $F_i^+ = (0, f_i]$. Let $F^N = \prod_{i \in N} F_i$ be the product set of the action spaces for players in $N$. For all $T \subseteq N$, a player-coalition $T \subseteq N$ corresponds in a canonical way to the fuzzy coalition $e^T \in F^N$, which is the vector with $e_i^T = 1$ if $i \in T$, and $e_i^T = 0$ if $i \in N \setminus T$. Denote $0_N$ the zero vector in $\mathbb{R}^N$.

A **fuzzy TU game** is a triple $(N, f, V)$, where $N$ is a non-empty and finite set of players, $f = (f_i)_{i\in N} \in [0,1]^N$ be the vector that describes the highest level of activity for each player, and $V : F^N \to \mathbb{R}$ is a characteristic function with $V(0_N) = 0$ which assigns to each action vector $\alpha = (\alpha_i)_{i\in N} \in F^N$ the worth that the players can obtain when each player $i$ plays at activity level $\alpha_i$. If no confusion can arise a game $(N, f, V)$ will sometimes be denoted by its characteristic function $V$. Let us consider a fuzzy TU game $(N, f, V)$ and $\alpha \in F^N$, we write $(N, \alpha, V)$ for the **fuzzy TU subgame** obtained by restricting $V$ to $\{\beta \in F^N \mid \beta_i \le \alpha_i$ , for all $i \in N\}$ only.

Denote the class of all fuzzy TU games by $\Gamma$. Let us consider $(N, f, V) \in \Gamma$, we define $L^{N,f} = \{(i, k_i) \mid i \in N, k_i \in F_i^+\}$. A **solution** on $\Gamma$ is a map $\psi$ assigning to each $(N, f, V) \in \Gamma$ an element

$$\psi(N, f, V) = \big(\psi_{i,k_i}(N, f, V)\big)_{(i,k_i)\in L^{N,f}} \in \mathbb{R}^{L^{N,f}}.$$

Here $\psi_{i,k_i}(N, f, V)$ is the power index or the value of the player $i$ when he takes action $k_i$ to play game $V$. For convenience, we define $\psi_{i,0}(N, f, V) = 0$ for each $i \in N$.

For $N \subseteq U$ and $\alpha \in \mathbb{R}^N$, let $S(\alpha) = \{i \in N |\ \alpha_i \ne 0\}$, $|T|$ be the number of elements in $T$ and $\alpha_T$ be the restriction of $\alpha$ at $T$ for each $T \subseteq N$. Let $i \in N$, for convenience we introduce the substitution notation $\alpha_{-i}$ to stand for $\alpha_{N\setminus\{i\}}$ and let $\beta = (\alpha_{-i}, t) \in \mathbb{R}^N$ be defined by $\beta_{-i} = \alpha_{-i}$ and $\beta_i = t$. Moreover, let $p \in N$ and $l \in \mathbb{R}$, $\alpha_{-ip}$ to stand for $\alpha_{N\setminus\{i,p\}}$ and $(\alpha_{-ip}, t, l)$ to stand for $((\alpha_{-i}, t)_{-p}, l)$.

In the framework of fuzzy TU games, Hwang and Liao (2009) proposed the consistent value as follows.

**Definition 1** *The* **consistent value** *of fuzzy games, $\gamma$, is the function on $\Gamma$ which associates to each $(N, f, V) \in \Gamma$, each player $i \in N$ and each $k_i \in F_i^+$ the value*

$$\gamma_{i,k_i}(N, f, V) = \sum_{\substack{S\subseteq S(f)\\ i\in S}} \frac{(|S|-1)!(|S(f)|-|S|)!}{|S(f)|!} \cdot \big[\, V\big((f_{-i}, k_i)_S, 0_{N\setminus S}\big) - V\big((f_{-i}, 0)_S, 0_{N\setminus S}\big)\big].$$

**Remark 1** *A fuzzy TU game, which is defined by Aubin (1974,1981), is a pair $(N, V^*)$, where $N$ is a coalition and $V^*$ is a mapping such that $V^* : [0,1]^N \longrightarrow \mathbb{R}$ and $V^*(0_N) = 0$. In fact, $(N, V^*) = (N, e^N, V)$ .*

# 3 Main Results

Let $N \subseteq U$, $i \in N$ and $x \in \mathbb{R}^N$, for convenience we introduce the substitution notation $x_{-i}$ to stand for $x_{N\setminus\{i\}}$ and let $y = (x_{-i}, j) \in \mathbb{R}^N$ be defined by $y_{-i} = x_{-i}$ and $y_i = j$.

For convenience, we will use lower case letter $v$ and capital letter $V$ to denote the characteristic functions of standard TU games and fuzzy TU games respectively. A **coalitional standard game with transferable utility (standard TU game)** is a pair $(N, v)$ where $N$ is a coalition and $v$ is a mapping such that $v : 2^N \longrightarrow \mathbb{R}$ and $v(\emptyset) = 0$. Denote the class of all standard TU games by $G$.

The **Shapley value (1953)**, denoted by $\phi$, is the solution on $G$ which associates with $(N, v) \in G$ and each player $i \in N$ the value

$$\phi_i(N, v) = \sum_{\substack{S\subseteq N\\ i\in S}} \frac{(|S|-1)!(|N|-|S|)!}{|N|!} \cdot \big[v(S) - v(S \setminus \{i\})\big]. \tag{1}$$

Without loss of generality, we can assume that $S(f) = N$. Let $(i, j) \in L^{N,f}$. We define the **TU decomposition game** $(N, v^{i,j})$ by

$$v^{i,j}(S) = V\big((f_{-i}, j)_S, 0_{N\setminus S}\big) \text{ for all } S \subseteq N. \tag{2}$$

Next, we state the main result as follows.

**Theorem 1** *Let $(N, f, V) \in \Gamma$, $(i, j) \in L^{N,f}$ and $(N, v^{i,j})$ be the TU decomposition game. Then*

$$\gamma_{i,j}(N, f, V) = \phi_i(N, v^{i,j}).$$

**Proof.** Let $(N, f, V) \in \Gamma$, $(i, j) \in L^{N,f}$ and $S \subseteq N \setminus \{\emptyset\}$ with $i \in S$. By Equation (2),

$$V\big((f_{-i}, j)_S, 0_{N\setminus S}\big) - V\big((f_{-i}, 0)_S, 0_{N\setminus S}\big) = v^{i,j}(S) - v^{i,j}(S \setminus \{i\}). \tag{3}$$

Combining Equation (3) with Equation (1) and Definition 1,

$$\begin{aligned}
\phi_i(N, v^{i,j}) &= \sum_{\substack{S\subseteq N\\ i\in S}} \tfrac{(|S|-1)!(|N|-|S|)!}{|N|!} \cdot \big[v^{i,j}(S) - v^{i,j}(S \setminus \{i\})\big]\\
&= \sum_{\substack{S\subseteq N\\ i\in S}} \tfrac{(|S|-1)!(|N|-|S|)!}{|N|!} \cdot \big[V\big((f_{-i}, j)_S, 0_{N\setminus S}\big) - V\big((f_{-i}, 0)_S, 0_{N\setminus S}\big)\big]\\
&= \gamma_{i,j}(N, f, V).
\end{aligned}$$

Hence, the proof is completed. ∎

Let $(N, v) \in G$ and $S \subseteq N$. The **unanimity games** for standard TU games $(N, v_S)$, where $S \subseteq N \setminus \{\emptyset\}$, defined by

$$v_S(T) = \begin{cases} 1 & \text{if } S \subseteq T, \\ 0 & \text{otherwise.} \end{cases}$$

It is known that for $(N, v) \in G$ it holds that $v = \sum_{S \subseteq N \setminus \{\emptyset\}} C_S \cdot v_S$, where $C_S = \sum_{T \subseteq S} (-1)^{|S|-|T|} v(T)$ is said to be the **dividend** among the players in $S$. It is known that the Shapley value also could be presented an alternative form as follows. For all $(N, v) \in G$ and for all player $i \in N$,

$$\phi_i(N, v) = \sum_{\substack{S \subseteq N \\ i \in S}} \frac{C_S}{|S|} \ . \tag{4}$$

**Corollary 1** *Let $(N, f, V) \in \Gamma$, $(i, j) \in L^{N,f}$ and $(N, v^{i,j})$ be the TU decomposition game. Then*

$$\gamma_{i,j}(N, f, V) = \sum_{\substack{S \subseteq N \\ i \in S}} \frac{C_S^{i,j}}{|S|} \ ,$$

*where $C_S^{i,j} = \sum_{T \subseteq S} (-1)^{|S|-|T|} v^{i,j}(T)$.*

**Proof.** This corollary follows by Theorem 1 and Equation (4). ■

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