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A formula for Nash equilibria in monotone singleton congestion games

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Abstract

This paper provides a simple formula describing all Nash equilibria in monotone symmetric singleton congestion games. Our approach also yields a new and short proof establishing the existence of a Nash equilibrium in this kind of congestion games.

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1. Introduction

Congestion games provide a natural framework for a wide range of economics and computer science applications such as resource allocation, routing and network design problems. Rosenthal (1973), who was the first to consider this class of noncooperative games, showed by a potential function argument, that they possess a pure-strategy Nash equilibrium. In fact, Nash dynamics, where players iteratively improve their utilities, always converge to an equilibrium after a finite number of steps. In Rosenthal's model (standard congestion games), a player's strategy consists of a subset of a common set of resources. The payoff received for selecting a particular resource depends only on the total number of players sharing this resource. The utility a player derives from a combination of resources is the sum of the payoffs associated with each resource included in his choice. Milchtaich (1996) introduced a new variant of congestion games, namely the (singleton) congestion games with player-specific payoff functions. In these games, each player has individual non increasing payoff functions and is allowed to choose only one resource and not several at a time. He showed that each game in this class admits at least one Nash equilibrium that can be rehashed as a terminal point of a particular improvement dynamic.

A substantial literature has been devoted to particular subclasses and extensions of congestion games. Most of the studies focus on the problem of finding and computing efficiently only one Nash equilibrium, leaving open the question of identifying all Nash equilibria. However, the characterization of the set of all equilibria, beyond its theoretical interest, can be very useful when we have to choose between these equilibria on the basis of performance criteria such as social optimality, or to explore intrinsic properties of the game such as the price of anarchy[1]. In this paper, we address this question for a simple subclass of congestion games which lie in the intersection between Rosenthal's and Milchtaich's model. We refer to games in this class as monotone symmetric singleton congestion games (SSCGs). Our approach yields a new and short proof establishing the existence of a Nash equilibrium in this kind of games and shows how to straightforwardly compute all equilibria using a simple and direct formula.

The rest of this document is organized as follows: section 2 briefly reviews the related work, section 3 establishes the result and section 4 concludes the paper.

2. Related work

The existence of Nash equilibria in SSCGs is a simple corollary of Rosenthal's theorem (1973), since this kind of games is a special case of standard congestion games. The family of SSCGs has been initially studied by Milchtaich (1996) as the symmetric case of his model. He showed that every game in the more general class of singleton congestion games with specific payoff functions possess Nash equilibria that can be computed in a polynomial number of steps (of a best-replay improvement path)[2]. Ieong *et al.* (2005) generalized this result to the largest class of singleton congestion games where the payoff functions are not required to be monotone. They also showed that even optimal Nash equilibria (for a certain class of optimality criteria) can be found in polynomial time. Holzman and Law-Yone (1997) and Voorneveld *et al.* (1999) investigated the set of strong

[1]When utilities are replaced by costs, the price of anarchy of a game is the ratio of the social cost in the worst Nash equilibrium to the minimum social cost possible.

[2]The class of congestion games with specific payoff functions was also studied independently by Quint and Shubik (1994) and by Konishi *et al.* (1997).

Nash equilibria in monotone singleton congestion games[3]. It turns out that this set coincides with the set of Nash equilibria and with the set of profiles which maximize the potential. Variants of (monotone) singleton congestion games have been studied in terms of time convergence of the best-reply dynamics to a Nash equilibrium (Even-Dar *et al.* 2003) and in terms of the existence of alternative concept of solution (Rozenfeld and Tennenholtz 2006).

Nevertheless, as was mentioned above, all these researches enable one to find only one (particular) equilibrium, ignoring the general structure of the set of all Nash equilibria. In the following, we improve the study of SSCGs by providing a simple formula describing all these equilibria. In order to state our result, we first need to simplify the analysis by moving to the ordinal representation of preferences, instead of the cardinal one (Milchtaich 1996). Indeed, for this kind of games, we can, without affecting the set of Nash equilibria, replace the values of the payment functions (i.e. cardinal representation) by their ranks in a preference ordering representing the common ordinal utility function.

3. The result

In our framework, a SSCG is represented by a tuple $\Gamma(N, R, \precsim)$ where N is a set of n players, R a set of m resources and $\precsim$ a weak ordering (a reflexive, transitive, antisymmetric and complete binary relation) on $R \times \{1, \ldots, n\}$. We assume that $\precsim$ is decreasing with the second component. This means that for all r in R and for all k and k' in $\{1, \ldots, n\}$, $k \leq k' \Rightarrow (r, k) \succsim (r, k')$. A player's strategy corresponds to the choice of a single resource in R. A (strategy) profile is an $n-$tuple $\sigma = (\sigma_i)_{i \in N}$ of R, where, for each i in N, σ_i denotes the strategy of player i. For a profile $\sigma = (\sigma_i)_{i \in N}$, let σ_{-i} be the same profile with i's strategy excluded, so that (σ_{-i}, σ_i) forms the complete profile σ. For a profile $\sigma = (\sigma_i)_{i \in N}$, the congestion on resource r (i.e. the number of players using r) is defined by $n_r(\sigma) = |\{i \in N : \sigma_i = r\}|$. The vector $(n_1(\sigma), \ldots, n_m(\sigma))$ is the congestion vector corresponding to σ. Note that players' preferences over strategy profiles depend only on the congestion on each resource: for player i, a profile σ is at least as good as a profile σ' if and only if $(\sigma_i, n_{\sigma_i}(\sigma)) \succsim (\sigma'_i, n_{\sigma'_i}(\sigma'))$. In this ordinal context, a strategy profile σ^* is a Nash equilibrium of the game Γ if σ^* is at least as good as $(\sigma_i, \sigma^*_{-i})$, for all i in N and all σ_i in R. Since players are anonymous, all strategy profiles that differ only by a permutation of players can be identified with the corresponding congestion vector. We refer to a congestion vector $\sigma^* = (n_1, \ldots, n_m)$ as a Nash equilibrium if no player can benefit from joining a (possibly empty) set of players sharing a different resource: for all r, r' in R with $r \neq r'$ and $n_r \geq 1$, we have $(r, n_r) \succsim (r', n_{r'} + 1)$.

In what follows, we develop a technique to enable a straightforward identification of all Nash equilibria in a SSCG. For this purpose we need the next definition.

Definition 1. *Let $\precsim$ be a weak ordering on $R \times \{1, \ldots, n\}$ decreasing with the second component. An n-sequence derived from $\precsim$ is a subset T of $R \times \{1, \ldots, n\}$ such that:*

- $|T| = n$.
- $((r, k) \in T$ *and* $(r', k') \notin T) \Rightarrow (r, k) \succsim (r', k')$.

[3]A strong Nash equilibrium is a profile for which no subset of players has a joint deviation that strictly benefits all of them, while all other players are expected to maintain their equilibrium strategies.

- $(r,k) \in T \Rightarrow ((r,k') \in T, \forall k' \in \{1,\dots,n\} : k' < k)$.

Thus, an $n-$sequence is simply a set of the most preferred n elements of $R \times \{1,\dots,n\}$. To illustrate this definition, let us consider the following situations.

Example 1.

- Let $N = \{1,2,3,4,5\}$ and $R = \{a,b,c\}$. For simplicity and without losing integrity, we will denote the couple (r,k) by kr. Suppose that the common ordinal utility function is given by the following strictly decreasing ordering:

  $$5c \prec 4c \prec 5a \prec 5b \prec 4b \prec 4a \prec 3a \prec 3b \prec 2b \prec 2a \prec \underbrace{3c \prec a \prec 2c \prec c \prec b}.$$

  By definition 1, the unique 5-sequence is $T = \{3c, a, 2c, c, b\}$.

- Let $N = \{1,2,3,4,5,6,7,8\}$ and $R = \{a,b,c,d\}$. Suppose that the players' preferences are given by the following weak ordering:
  $8c \sim 8b \prec 8a \sim 8d \prec 7c \sim 7b \sim 6c \prec 7d \sim 5c \sim 4c \prec 3c \sim 6b \sim 6d \prec 5d \sim 5b \sim 4b \sim 7a \prec 6a \sim 4d \sim 5a \prec 4a \sim 3a \sim 3b \sim 2b \sim 3d \prec 2a \sim b \sim 2c \prec c \sim 2d \prec a \sim d$.
  We have exactly three 8-sequences: $T_1 = \{3d, 2a, b, 2c, c, 2d, a, d\}$, $T_2 = \{2b, 2a, b, 2c, c, 2d, a, d\}$ and $T_3 = \{3a, 2a, b, 2c, c, 2d, a, d\}$.

We can now formulate our result.

Theorem 1. *Let $\Gamma(N, R, \precsim)$ be a monotone symmetric singleton congestion game, with $|N| = n$ and $|R| = m$. Then,*

1. *To each n-sequence T of $\precsim$ corresponds a unique Nash equilibrium of Γ. This equilibrium is defined by $\sigma^* = (\alpha_1, \dots, \alpha_m)$, where α_r is defined by, for $r = 1,\dots,m$,*

   $$\alpha_r = \begin{cases} \max\{p : (r,p) \in T\} & \text{if } (r,1) \in T \\ 0 & \text{otherwise.} \end{cases}$$

   Reciprocally, each Nash equilibrium of the game Γ corresponds to an n-sequence of $\precsim$.

2. *When the players' preferences are expressed by a strictly decreasing ordering, the game Γ admits exactly one Nash equilibrium.*

3. *The number of Nash equilibria of the game Γ equals the number of all n-sequences extracted from $\precsim$.*

Proof. Since 2. and 3. are simple consequences of 1., it simply remains to prove the first assertion. Let T be an n-sequence and let $\sigma^* = (\alpha_1, \dots, \alpha_m)$ be the $m-$components vector defined by $\alpha_r = \max\{p : (r,p) \in T\}$ if $(r,1) \in T$ and $\alpha_r = 0$ if $(r,1) \notin T$. It is clear that $\sum_{r=1}^{m} \alpha_r = n$. Indeed, we can write $R = R_0 \cup R_1$, where $R_0 = \{r \in R : (r,1) \notin T\}$ and $R_1 = \{r \in R : (r,1) \in T\}$. Let $r_1, \dots, r_t$ be an enumeration of R_1 $(1 \le t \le m)$. We have $\sum_{r=1}^{m} \alpha_r = \sum_{r \in R_0} \alpha_r + \sum_{r \in R_1} \alpha_r$, $\sum_{r \in R_0} \alpha_r = 0$ and $\sum_{r \in R_1} \alpha_r = n$, since the sequence T is exclusively constituted by the terms $(r_1, \alpha_{r_1}), \dots, (r_1, 1), (r_2, \alpha_{r_2}), \dots, (r_2, 1), \dots, (r_t, \alpha_{r_t}), \dots, (r_t, 1)$. Therefore, $\sum_{r=1}^{m} \alpha_r = n$, whence σ^* is a congestion vector. Furthermore, for all r, r' in R such that $\alpha_r \geq 1$, we have $(r, \alpha_r) \succsim (r', \alpha_{r'} + 1)$ because $(r, \alpha_r) \in T$ and $(r', \alpha_{r'} + 1) \notin T$. Hence, σ^* is a Nash equilibrium.

Reciprocally, let $\sigma^* = (\alpha_1, \dots, \alpha_m)$ be a Nash equilibrium. Let $r_1, \dots, r_t$ be an enumeration of the set R_1 defined by $R_1 = \{r \in R : \alpha_r \geq 1\}$ and let $T = \{(r_1, \alpha_{r_1}), \dots, (r_1, 1), \dots, (r_t, \alpha_{r_t}), \dots, (r_t, 1)\}$. Obviously, T is an n-sequence. In fact, as σ^* is a congestion vector, we have $\sum_{r=1}^{m} \alpha_r = n$ and so $|T| = n$. On the other hand, by definition of T, $(r, k) \in T \Rightarrow ((r, k') \in T, \forall k' \in \{1, \dots, n\}$ such that $k' < k)$. Finally, let $(r, k) \in T$ and $(r', k') \notin T$. By definition of T, we have $k \leq \alpha_r$ and $k' \geq \alpha_{r'} + 1$. Since σ^* is a Nash equilibrium, we have $(r, \alpha_r) \succsim (r', \alpha_{r'} + 1)$. By the monotonicity hypothesis, we obtain $(r, k) \succsim (r, \alpha_r) \succsim (r', \alpha_{r'} + 1) \succsim (r', k')$. □

To illustrate the above theorem, we continue with the previous example to show how we can easily characterize all Nash equilibria.

Example 2. Reconsider the two cases of the first example. Applying our theorem, we can find a Nash equilibrium for each n-sequence.

- Let $N = \{1, 2, 3, 4, 5\}$ and $R = \{a, b, c\}$. Considering the players' ordinal utility: $5c \prec 4c \prec 5a \prec 5b \prec 4b \prec 4a \prec 3a \prec 3b \prec 2b \prec 2a \prec \underbrace{3c \prec a \prec 2c \prec c \prec b}$, we obtain $T = \{3c, a, 2c, c, b\}$. Selecting the greatest integer corresponding to each resource, we identify the "unique" Nash equilibrium: $\sigma^* = (1, 1, 3)$. This means that all Nash equilibria for this game correspond to the unique congestion vector defined by $n_a = 1$, $n_b = 1$ and $n_c = 3$. Thus, a profile is a Nash equilibrium of this game if and only if it is a permutation of the profile (a, b, c, c, c). For simplicity, we note $\sigma^* = (a, b, 3c)$.

- Let $N = \{1, 2, 3, 4, 5, 6, 7, 8\}$ and $R = \{a, b, c, d\}$ and the weak ordering : $8c \sim 8b \prec 8a \sim 8d \prec 7c \sim 7b \sim 6c \prec 7d \sim 5c \sim 4c \prec 3c \sim 6b \sim 6d \prec 5d \sim 5b \sim 4b \sim 7a \prec 6a \sim 4d \sim 5a \prec 4a \sim 3a \sim 3b \sim 2b \sim 3d \prec 2a \sim b \sim 2c \prec c \sim 2d \prec a \sim d$.
Here we find one Nash equilibrium per n-sequence:

  For $T_1 = \{3d, 2a, b, 2c, c, 2d, a, d\}$, $\sigma_1^* = (2a, b, 2c, 3d)$;
  For $T_2 = \{2b, 2a, b, 2c, c, 2d, a, d\}$, $\sigma_2^* = (2a, 2b, 2c, 2d)$;
  For $T_3 = \{3a, 2a, b, 2c, c, 2d, a, d\}$, $\sigma_3^* = (3a, b, 2c, 2d)$.

  Hence, there are exactly three Nash equilibria in this game.

Note that the method described by Theorem 1 is not appropriate to the non-symmetric case, where players are restricted to choose only one strategy, but they each have their own utility function. The following example illustrates this fact.

Example 3. Let $N = \{1, 2, 3\}$ and $R = \{a, b, c\}$. We consider the following players' ordinal preferences:

$$3a \prec_1 3b \prec_1 2a \prec_1 3c \prec_1 2b \prec_1 a \prec_1 b \prec_1 2c \prec_1 c.$$

$$3c \prec_2 2c \prec_2 3b \prec_2 c \prec_2 2b \prec_2 3a \prec_2 b \prec_2 2a \prec_2 a.$$

$$3c \prec_3 3a \prec_3 2a \prec_3 2c \prec_3 3b \prec_3 c \prec_3 2b \prec_3 b \prec_3 a.$$

The concept of an $n-$sequence does not apply in this case because we have three different orderings. For player 1, we have the 3-sequence $b \prec_1 2c \prec_1 c$, for player 2: $b \prec_2 2a \prec_2 a$ and for player 3: $2b \prec_3 b \prec_3 a$. Applying Theorem 1 to these sequences, we obtain the following three congestion vectors: $(b, 2c)$, $(2a, b)$ and $(a, 2b)$. However, none of these three congestion vectors is appropriate to the three players simultaneously. Thus, we could think about taking the last term of each of the three above orderings. In this way, the strategy profile would be (c, a, a). But one can easily check that this profile does not correspond to a Nash equilibrium. Nevertheless, there exists a Nash equilibrium which is the profile (c, a, b).

4. Concluding remarks

In this paper we have proposed a new approach which enable one to find all Nash equilibria of a given symmetric singleton congestion game. While we do not deal with the question of the computational complexity, we believe that our formula can contribute to the algorithmic analysis of this class of congestion games. For example, it can help to improve the time complexity of computing optimal Nash equilibria or calculate the price of anarchy. In future research, we hope to extend our approach to the general case of non-symmetric congestion games with player-specific payoff functions.

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