Economics Bulletin

Volume 33, Issue 1

The Kemeny rule and committees elections

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Abstract

An adaptation of the Kemeny rule (Kemeny 1959) was proposed by Ratliff (2003) for committees elections. A Committee is a fixed-size subset of candidates. Ratliff (2003) showed that the elected committee under the rule he proposed is not always made of the top candidates of the Kemeny ranking. We show that when restricting the frame to three-candidate elections, the elected committee of two candidates is always made of the two top candidates of the Kemeny ranking.

Citation: Eric Kamwa, (2013) "The Kemeny rule and committees elections", *Economics Bulletin*, Vol. 33 No. 1 pp. 648-654.
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Submitted: December 05, 2012. **Published:** March 12, 2013.

1 Introduction

Given an election with at least three candidates and the preferences of voters, the Kemeny rule (Kemeny 1959, Kemeny and Snell 1960) leads to a transitive ranking of the candidates. It operate by computing the distances from a given linear order to all the linear orders of the preferences profile. The Kemeny ranking is the linear order that minimizes the total distance to the whole profile and the *Kemeny winner* is the candidate at the top of this ranking. The Kemeny ranking is called the *consensual ranking*or the *ranking of compromise*[1] or the *median preference ordering*[2]. The Kemeny rule belongs to the family of the *Condorcet Consistent rules*. A voting rule is *Condorcet Consistent* if it always elects the Condorcet winner (a candidate that beats all the others in pairwise majority) when it exists.

According to Felsenthal (2012), when electing a unique winner, the Kemeny rule appears among the Condorcet Consistent rules as one of the most desirable rules. As the Kemeny rule produces a complete transitive ranking on candidates, if the social objective was to elect more than one winner, one could say that the winners will be the top ranked candidates of the Kemeny ranking. So, if the objective is to elect a committee or a board of fixed-size g ($g \geq 2$), this committee will be made by the g first candidates of the Kemeny ranking. Many authors do not agree with this way of doing.

Dodgson (1876, 1885) pointed out that some inconsistencies can raise when electing committees or a subsets of at least two candidates. For example, an elected committee can have as a member, a candidate such that there is one or more candidates outside that defeat him in pairwise majority. Even worse, all the committee members are majority dominated. Following this remark, Barberà and Coelho (2008) claimed that choose a non-controversial fixed-size set of candidates is to choose the set such that none of its member is majority dominated. A such set is called the *Weak Condorcet Set* or *Weak Condorcet Committee*. This concept was first introduced by Gehrlein (1985). A more restrictive version of this concept, the *Condorcet committee*, is analyzed in Ratliff (2003). The *Condorcet committee* is a fixed-size set of candidates such that every candidate in its majority dominates every candidate outside (see also Good 1971, and Miller 1980 for related topics.).

Ratliff (2003) designed an adaptation of the Kemeny rule for committee elections that that always selects the *Condorcet committee* when it exists[3]. This adaptation of the Kemeny rule (hereafter, KE rule) selects the set of g candidates with the smallest total margin of loss in pairwise majority versus the $m - g$ remaining candidates. Also, the KE rule always select the weak Condorcet Committee when there is one. Ratliff (2003) showed that an elected committee under the KE rule can be made of the g bottom candidates of the Kemeny ranking. The example used by Ratliff (2003) involves more than three candidates. Here, we show that, when restricting the voting frame to three-candidate elections, we have positive a result : when electing a committee of two members among three candidates, this committee is always made of the two top candidates of the Kemeny ranking.

The rest of the paper is organized as follows : section 2 is devoted to the basic notations and definitions. Section 3 presents our mains results and Section 4 concludes.

[1]See Young (1995).

[2]This means that this ordering is the one from which the sum of absolute deviations (or absolute distances) of all the voters rankings is minimized. (see Felsenthal 2012).

[3]When it exists, the *Condorcet committee* is unique (see Good 1971).

2 Notation and definitions

2.1 Preferences

Let N be the set of n voters ($n \geq 2$) and A the set of m candidates, $m \geq 3$. The binary relation R over A is a subset of the cartesian product $A \times A$. For $a, b \in A$, if $(a, b) \in R$, we note aRb to say "a is at least good as b". $\neg aRb$ is the negation of aRb. If we have aRb and $\neg bRa$, we will say "a is better or strictly preferred to b". In this case, we write aPb with P the asymmetric component of R. The symmetric component of R, I, is defined by aIb translating an an indifference between a and b *i.e* $\neg aRb$ and $\neg bRa$. The preference profile $\pi = (P_1, P_2, ..., P_i, ..., P_n)$ gives all the linear orders[4] of all the n voters on A where P_i is the strict ranking of a given voter i. The set of all the preference profiles of size n on A is denoted by $P(A)^n$. We will simply write abc to say that a is strictly preferred to b who is strictly preferred to c. A voting situation $\tilde{n} = (n_1, n_2, ..., n_t, ..., n_{m!})$ indicates the number of voters for each linear order such that $\sum_{t=1}^{m!} n_t = n$. Table 2.1 gives the voting situation with three candidates.

Table 2.1: voting situation with $A = \{a, b, c\}$

$n_1 : abc$	$n_2 : acb$	$n_3 : cab$	$n_4 : cba$	$n_5 : bca$	$n_6 : bac$

If the number of voters who rank a before b is greater than that of those who b before a, a is said majority preferred to b. We denote this by $aM(\pi)b$ or simply aMb when there is no ambiguity. Candidate a is the *Condorcet winner* if we have $aM(\pi)b$ for all $b \in A \setminus \{a\}$. Candidate a is the *Condorcet loser* if we have $bM(\pi)a$ for all $b \in A \setminus \{a\}$. We denote by n_{ab} the number of voter that prefer candidate a to candidate b. If aMb, we say that candidate b losses the pairwise by a margin equal to $n_{ab} - n_{ba}$.

Suppose that we want to elect a committee of size g ($2 \leq g \leq m - 1$). We denote by C^g the set of all possible committees of size g. A *Condorcet committee* is a fixed size subset of candidates such that all candidate in this subset defeats all outside candidate in pairwise majority. A *weak Condorcet committee* is a fixed size subset of candidates such that no candidate in this subset is defeated by any outside candidate in pairwise majority.

Definition 1. (*Condorcet committee*). With $\sharp A = m$, $C \in C^g$ is a Condorcet committee if and only if $\forall x \in C$ we have xMy for all $y \in A \setminus C$.

Definition 2. (*Weak Condorcet committee*). With $\sharp A = m$, $C \in C^g$ is a weak Condorcet committee if and only if $\forall x \in C$ we have yMx for no $y \in A \setminus C$.

For a given voting situation, a (weak) Condorcet committee may not exist given g (see Good 1971, and Gehrlein 1985). In a voting situation, a Condorcet committee is also a weak Condorcet committee. The reverse is not true.

[4]A linear order is a binary relation that is transitive, complete and antisymmetric. The binary relation R on A is *transitive* if for $a, b, c \in A$, if aRb and bRc then aRc. R is *antisymmetric* if for all for $a \neq b$, $aRb \Rightarrow \neg bRa$; if we have aRb and bRa, then $a = b$. R is *complete* if and only if for all $a, b \in A$, we have aRb or bRa.

Table 2.2: The Kemeny distances for the profile of table 2.1

$K(\pi, abc) = 2n_2 + 4n_3 + 6n_4 + 4n_5 + 2n_6$
$K(\pi, acb) = 2n_1 + 2n_3 + 4n_4 + 6n_5 + 4n_6$
$K(\pi, cab) = 4n_1 + 2n_2 + 2n_4 + 4n_5 + 6n_6$
$K(\pi, cba) = 6n_1 + 4n_2 + 2n_3 + 2n_5 + 4n_6$
$K(\pi, bca) = 4n_1 + 6n_2 + 4n_3 + 2n_4 + 2n_6$
$K(\pi, bac) = 2n_1 + 4n_2 + 6n_3 + 4n_4 + 2n_5$

2.2 The Kemeny rule and the KE rule

Given two candidates $a, b \in A$ and two linear orders P_i, P_j, the Kemeny metric $\delta(P_i, P_j)$ between P_i and P_j is the number of time that the relative ranking on two candidates differs in P_i and P_j. Given a linear order preference P and $\pi = (P_1, P_2, ..., P_i, ..., P_n)$, the Kemeny distance between $K(\pi, P)$, P and π is given by:

$$K(\pi, P) = \sum_{i \in N} \delta(P, P_i)$$

For the voting situation of table 2.1, the Kemeny distances of each of the six possible linear orders are given in table 2.2.

The Kemeny ranking $K(\pi)$ is the linear order P that minimizes $K(\pi, P)$ and the Kemeny winner $K(\pi, A)$ is the candidate top ranked in $K(\pi)$.

Definition 3. Given $P \in P(A)^n$, $K(\pi) = P$ if

$$K(\pi, P) \leq K(\pi, P'),\ \forall P' \in P(A)^n$$

Given a profile π and C^g the set of committees of size g, the KE-score $KE(\pi, C)$ of a committee $C \in C^g$ is the total lost margin of candidates in C in pairwise majority versus candidates outside of C. So,

$$KE(\pi, C) = \sum_{x \in C, y \in A \setminus C} \max[0, n_{yx} - n_{xy}]$$

Definition 4. Given a profile π with $\sharp A = m \geq 3$ and C^g the set of all possible committees of size g. Respectively, the KE outcome set is defined as follows

$$\mathcal{KE}_g(\pi) = \{C \in C^g : KE(\pi, C) \leq KE(\pi, C')\ \ \forall C' \in C^g \setminus C\}$$

For the voting situation of table 2.1, with $C^2 = \{(a, b), (a, c), (b, c)\}$, the KE scores are are given in table 2.3.

Let us take an example to give an illustration of the KE rule.

Example 1. *Consider the following voting situation with three candidates and 22 voters.*

voters preferences

$5 : abc$	$2 : acb$	$3 : cab$	$1 : cba$	$8 : bca$	$3 : bac$

Table 2.3: The KE scores for $C^2 = \{(a,b),(a,c),(b,c)\}$

$KE(\pi,(a,b)) = \max[0, n_{ca} - n_{ac}] + \max[0, n_{cb} - n_{bc}]$
$KE(\pi,(a,c)) = \max[0, n_{ba} - n_{ab}] + \max[0, n_{cb} - n_{bc}]$
$KE(\pi,(b,c)) = \max[0, n_{ab} - n_{ba}] + \max[0, n_{ac} - n_{ca}]$

After computations, we get $K(\pi, abc) = 60$, $K(\pi, acb) = 80$, $K(\pi, cab) = 76$, $K(\pi, cba) = 72$, $K(\pi, bca) = 52$ *and* $K(\pi, bac) = 56$. *So, the consensual ranking is bca. With the KE rule, we have* $\mathcal{KE}_2(\pi) = \{(b,c)\}$ *since* (b,c) *is the Condorcet Committee. In this voting situation the elected committee of two members under the KE rule is formed by the two top candidates of the Kemeny ranking.*

3 Results

As shown in example 1, the elected committee of two members under the KE rule is made of the two top ranked candidates of the Kemeny ranking. In Ratliff (2003, p442) an example with more than three candidates is provided in which the elected candidates of the KE rule are those who are bottom ranked in the Kemeny ranking. Our main result tells us that when a voting situation involves only three candidates, the two-member committees elected with the KE rule always consists of the two top ranked candidates of the kemeny rule.

Proposition 1. *In three-candidate elections, The Kemeny committee of two members is always made of the two top ranked candidates of the Kemeny ranking.*

Proof. For the proof, let us analyze all the possible configurations. In three-candidate elections, the KE scores are based on the pairwise comparisons. For the voting situation of table 2.1, the pairwise majority can lead to one of the following configurations[5]:

No	configurations
(1)	$aM(\pi)b$, $aM(\pi)c$ and $bT(\pi)c$
(2)	$aM(\pi)c$, $bM(\pi)c$ and $aT(\pi)b$
(3)	$aM(\pi)b$, $aM(\pi)c$ and $bM(\pi)c$
(4)	$aM(\pi)b$, $bM(\pi)c$ and $cM(\pi)a$
(5)	$aM(\pi)c$, $cM(\pi)b$ and $bM(\pi)a$
(6)	$aT(\pi)b$, $aT(\pi)c$ and $bT(\pi)c$

Let's analyze each of these configurations.

- In configuration (1), there is a Condorcet winner (candidate a) and there are two weak Condorcet committees (a,b) and (a,c). Thus, we always have $\mathcal{KE}_2(\pi) = \{(a,b),(a,c)\}$. Since candidate a is the Condorcet winner, he is always ranked first in the Kemeny ranking. So, the Kemeny ranking will be *abc* or *acb*: no matter what is the Kemeny ranking, the two top candidates always belong to the two-member committee elected under the KE-rule. Configuration (1) always lead to an agreement.

[5] Neutrality is assumed : permute the name of candidates does not matter.

- With configurations (2) and (3), candidate c is the Condorcet loser and $\{(a,b)\}$ is the Condorcet committee; so, $\mathcal{KE}_2(\pi) = \{(a,b)\}$. One can check that the Kemeny ranking is certainly abc in configuration (3) and abc or bac in configuration (2). Thus, the two-member committee selected coincides with two top ranked candidates of the Kemeny ranking.

- Since Configurations (4) and (5) describe a majority cycle among candidates, they are symmetric. In each of these configurations, there is no (weak) Condorcet committee of size two. Consider Configuration (4) and assume that abc the is the Kemeny ranking such that $\{(a,b)\} \notin \mathcal{KE}_2(\pi)$. By definition, $\{(a,b)\} \notin \mathcal{KE}_2(\pi)$ implies $KE(\pi,(a,b)) > KE(\pi,(a,c))$ (i) and/or $KE(\pi,(a,b) > KE(\pi,(b,c)$ (ii). Since there is no (weak) Condorcet committee, $KE(\pi,(a,b)) > 0$, $KE(\pi,(a,c)) > 0$ and $KE(\pi,(b,c)) > 0$. According to table 2.3, we have what follows:

$$\text{by (i)},\quad KE(\pi,(a,b)) - KE(\pi,(a,c)) > 0 \quad\Leftrightarrow\quad n_4 + n_5 > n_1 + n_2$$
$$\text{by (ii)},\quad KE(\pi,(a,b)) - KE(\pi,(b,c)) > 0 \quad\Leftrightarrow\quad n_3 + n_4 > n_1 + n_6$$

  Also, if abc the is the Kemeny ranking, this implies among others that $K(\pi,abc) < K(\pi,cab)$ and $K(\pi,abc) < K(\pi,bca)$. Using table 2.2, the reader can check that $K(\pi,abc) - K(\pi,cab) < 0 \Leftrightarrow n_3 + n_4 < n_1 + n_6$ (iv) and $K(\pi,abc) - K(\pi,bca) < 0 \Leftrightarrow n_4 + n_5 < n_1 + n_2$ (v). It comes that (iv) contradicts (ii) and (v) contradicts (i). Thus, if abc the is the Kemeny ranking, we certainly have $\{(a,b)\} \in \mathcal{KE}_2(\pi)$.

- In Configuration (6), all the candidates tie. So, $\mathcal{KE}_2(\pi) = \{(a,b),(a,c),(b,c)\}$. Also, all the six possible linear orders have the same Kemeny score. Similarly to Configuration (1), no matter which linear order is chosen as the collective ranking, the two top candidates always belong to $\mathcal{KE}_2(\pi)$. With Configuration (6), the KE rule always elects the two top candidates of the Kemeny ranking.

□

4 Conclusion

We have shown that elect a committee of two members among three candidates with the adapted Kemeny rule proposed by Ratliff (2003) (the KE rule) is equivalent to select the two top candidates of the Kemeny ranking. With more candidates, this is no more the case (as shown by Ratliff 2003) no matter the size of the committee to be elected. Thus, the three-candidate elections is the only case for which the KE rule always agrees with the top ranked candidate of the Kemeny rule. What could be done next, is to find, in four-candidate elections, the probability of discordance between the KE rule and the Kemeny ranking when electing a committee of two or three members. This is a hard an cumbersome job that can be tackled with the new tools available in the social choice literature.

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