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Responsiveness axioms and the majority rule

Adrian Miroiu
SNSPA

Abstract

The responsiveness property is an essential ingredient of May's (1952) famous axiomatization of the (simple) majority rule. However, it comes in more variants: May's classic property, Additive Responsiveness and Independence of Indifferent Societies. In this note it is proved that although the three properties can be used to produce very similar axiomatizations of the majority rule, they are independent of each other and not equally strong.

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Contact: Adrian Miroiu - admiroiu@snspa.ro.
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1. Introduction

The responsiveness property is an essential ingredient of May's (1952) famous axiomatization of the (simple) majority rule μ. It was intended to convey the idea that each individual can be pivotal in a group decision. However, some authors rejected responsiveness and tried to present characterizations of μ that do not appeal to it (Dasgupta and Maskin: 2008; Campbell and Kelly: 2000; Aşan and Sanver: 2002). Others tried to use variants of it to uncover the structure of the set of defining properties of the majority rule (Miroiu: 2004, Woeginger: 2005, Quesada: 2011).

This paper discusses three different responsiveness properties: May's classic property (**MR**), Additive Responsiveness (**AR**) and Independence of Indifferent Societies (**IIS**). It is proved that the three properties can be used to produce very similar axiomatizations of μ. However, the three responsiveness properties are independent of each other. Moreover, they are not equally strong: **IIS** is shown to be weaker than both **MR** and **AR**; and **AR** is shown to be weaker than **MR**.

2. The framework

Let N be a finite non-empty set. The members of N designate individuals. A society (or a group) is a non-empty subset of N. The set of alternatives is $\{x, y\}$, with $x \neq y$. A preference profile is a function p_N: $N \rightarrow \{-1, 0, 1\}$ assigning an individual preference $p_N(i)$ to each member $i \in N$. If the number is 1, x is preferred by i to y; if -1, y is preferred by i to x; if 0, i regards x as indifferent to y. **P** denotes the set of all preference profiles. For each preference profile p_S and society $S \subseteq N$, the restriction of p_N to S is denoted by p_S. So, the profile p_S of a society S is determined by the profile p_N of N. For each society S, the set of all restricted preference profiles p_S is denoted by $\mathbf{P}_S$. Let $-p_S$ denote the profile $p_S^{'}$ with the property that $p_S^{'}(i) = -p_S(i)$ for all $i \in S$. For any two profiles $p_S^{'}$ and $p_S^{''}$ of a society S, write $p_S^{'} \geq p_S^{''}$ if for all voters $i \in S$ we have that $p_S^{'}(i) \geq p_S^{''}(i)$; and write $p_S^{'} > p_S^{''}$ if if for all voters $i \in S$ we have that $p_S^{'}(i) \geq p_S^{''}(i)$ and $p_S^{'}(i) > p_S^{''}(i)$ for some $i \in S$.

A social welfare function is a mapping f: $\bigcup_{S \subseteq N} \mathbf{P}_S \rightarrow \{-1, 0, 1\}$. For each profile p_S of a society S, f gives the collective preference of its members over the alternatives x and y. The majority rule is the social welfare function μ: $\bigcup_{S \subseteq N} \mathbf{P}_S \rightarrow \{-1, 0, 1\}$ such that for all $p_S \in \mathbf{P}_S$, $\mu(p_S) = \text{sgn}(\sum_{i \in S} p_i)$, where the sgn function is defined by: (i) if $n > 0$, then sgn(n) = 1; (ii) if $n < 0$, then sgn(n) = -1; and (iii) if $n = 0$, then sgn(n) = 0.

The following three standard properties will be used in what follows.

Weak Pareto (**WP**): If $p_S(i) = 1$ for all $i \in S$, then $f(p_S) = 1$.

Neutrality (**N**). For each society S and each profile p_S, $f(p_S) = -f(-p_S)$.

Anonymity (**A**). For any two profiles $p_S^{'}$ and $p_S^{''}$, if the preferences in $p_S^{'}$ are a permutation of the preferences in $p_S^{''}$, then $f(p_S^{'}) = f(p_S^{''})$.

The responsiveness axiom comes in more than one variant. **MR** is the original May's (1952) condition. **AR** is its additive counterpart, explicitly introduced in Miroiu (2004). **IIS** was used in Quesada (2011). By May's classic property **MR** if the society is not against an alternative, and a single voter in it becomes more favorable to that alternative, then the society must strictly prefer it. **AR** states that if a society *S* is not against an alternative, and a voter who strictly prefers that alternative is added to *S*, then the new society will follow this voter. So **AR** requires that new voters be taken into account, rather than letting voters change their minds. By **IIS**, if a new society is added to an indifferent society, then the preference of the resulting society will follow the preference of the new society (or, to put it differently, indifferent subgroups do not count in the aggregate preference).

May Responsiveness (**MR**). If $p_S^{'} < p_S^{''}$ and $f(p_S^{'}) \geq 0$, then $f(p_S^{''}) = 1$.

Additive responsiveness (**AR**). Let $j \notin S$ and $p_N(j) = 1$.Then $f(p_S) \geq 0$ entails $f(p_{S\cup\{j\}})$ = 1.

Independence of Indifferent Societies (**IIS**). If *S* and *S′* are two societies that do not overlap (i.e., $S \cap S' = \varnothing$) and $f(p_{S'}) = 0$, then $f(p_{S \cup S'}) = f(p_S)$.

3. Axiomatizations

How are the three responsiveness properties connected? The following proposition gives a first answer:

Proposition 1. The axioms **IIS**, **MR** and **AR** are independent of each other.

Proof. I define three social welfare functions and show that each of them satisfies exactly one axiom.

a) For **IIS**, *f* is simply the constant function $f(p_S) = 0$ for all *S*. It satisfies **IIS**, but neither **MR** nor **AR**. Take $S = \{i\}$. For $p_N(i) = 0$, we have $f(p_{\{i\}}) = 0$. But if the preference of *i* changes to $p_N(i) = 1$, **MR** requires that $f(p_{\{i\}}) = 1$ – in contradiction with the definition of *f*. Since $f(p_{\{i\}}) = 0$, if $p_N(j) = 1$ then **AR** entails that $f(p_{\{i,j\}}) = 1$ – again in contradiction with the definition of *f*.
b) For **MR**, the function *f* is defined as follows: if $S = \{i\}$, then $f(p_{\{i\}}) = p_N(i)$; if $|S| \geq 2$, then $f(p_S) = -1$. To see that *f* is satisfied by **MR** is suffices to take into account only societies with exactly one member. We can easily check that **MR** is satisfied by *f* in this case. Now consider the society $\{i\}$. If $p_N(i) \geq 0$, then $f(p_{\{i\}}) \geq 0$; but if $p_N(j) = 1$ then by **AR** we get that $f(p_{\{i,j\}}) = 1$, while by the definition of *f* we must have $f(p_{\{i,j\}})$ = -1. So *f* does not satisfy **AR**. To show that *f* violates **IIS**, let $p_N(i) = 0$ and $p_N(j) = 1$. Then $f(p_{\{i\}}) = 0$ and $f(p_{\{j\}}) = 1$. In this case **IIS** entails that $f(p_{\{i,j\}}) = 1$. But by definition $f(p_S) = -1$ – contradiction.
c) For **AR**, the function *f* is defined by: $f(p_{\{i\}}) = 1$ if $p_N(i) = 1$, and $f(p_{\{i\}}) = 0$ if $p_N(i) = 0$ or $p_N(i) = -1$. If $|S| \geq 2$, then $f(p_S) = 1$. The function *f* satisfies **AR**. But if $p_N(i) = p_N(j)$ = -1, then $f(p_{\{i\}}) = 0$ and **IIS** entails that $f(p_{\{i,j\}}) = f(p_{\{j\}}) = 0$ – contradiction. So *f* does not satisfy **IIS**. Moreover, *f* does not satisfy **MR**. Suppose that $p_N(i) = -1$; then $f(p_{\{i\}})$

= 0. Now suppose that the individual i becomes more favorable to the alternative x so that $p_N(i) = 0$. But by the definition of f we still have $f(p_{\{i\}}) = 0$ – contradiction.

The three responsiveness axioms[1] can be used to produce very similar axiomatizations of the majority rule. Theorem 1 presents such results:

Theorem 1. A social welfare function f is the majority rule μ if and only if:

a) f satisfies **A**, **N** and **MR**; or
b) f satisfies **A**, **N** and **AR**; or
c) f satisfies **A**, **N**, **IIS** and **WP**.

Part (a) was proved in May (1952). Part (b) was proved in Woeginger (2005). I shall prove part (c) of the theorem. For the $\Longrightarrow$ direction, it can be easily checked that the majority rule μ satisfies all the four properties **A**, **N**, **IIS** and **WP**. For the converse $\Longleftarrow$ direction, I shall prove two auxiliary propositions. In conjunction with part (b) of the theorem, they immediately entail (c).

Proposition 2.

a) If the social welfare function f satisfies **WP** and **N**, then $f(p_{\{i\}}) = p_N(i)$.
b) If the social welfare function f satisfies **WP**, **N** and **IIS** and $f(p_S) = 1$, then there is some $j' \in S$ such that $p_N(j') = 1$.
c) If the social welfare function f satisfies **WP**, **N**, **A** and **IIS** and $f(p_S) = 1$ and $p_N(i) = 1$ for some $i \in S$, then $f(p_{S - \{i\}}) \geq 0$.

Proof. Expression (a) expresses the intuitive property that if a society consists in just one member, then its preference must be determined by this member. Let us consider the three possible cases. If $S = \{i\}$ and $p_N(i) = 1$, then $f(p_{\{i\}}) = 1 = p_N(i)$ by **WP**. If $p_N(i) = -1$, then $f(p_{\{i\}}) = -1 = p_N(i)$ by **N**. If $p_N(i) = 0$, then **N** yields that $f(p_S) = -f(-p_S) = 0$.

For (b), suppose that $f(p_S) = 1$ but $p_N(j') \neq 1$ for all $j' \in S$. We may distinguish three subcases:

- For all j' in S, $p_N(j') = 0$. Since **N** holds, we have that $f(p_S) = 0$, in contradiction with our supposition.
- For all j' in S, $p_N(j') = -1$. By **WP** we have that $f(p_S) = 1$ if for all j' in S it is the case that $p_N(j') = 1$. Then **N** immediately gives that $f(p_S) = -1$ – contradiction.
- For all j' in S, $p_N(j') \leq 0$ and $p_N(j'') = -1$ for some $j'' \in S$. Let S' be the set of all the members j' of S such that $p_N(j') = 0$. As proved above, we have $f(p_{S'}) = 0$. Then by **IIS** we have: $f(p_S) = f(p_{S - S'})$. But for all members j'' of $S - S'$ we have $p_N(j'') = -1$ an so $f(p_{S - S'}) = -1$ by **N** and **WP** – contradiction.

The proposition (c) is proved by induction on the number of members of the society S. If $|S| = 2$, then put $S = \{i, j\}$. Let $f(p_S) = 1$ and $p_N(i) = 1$. Suppose that $f(p_{S - \{i\}}) = f(p_{\{j\}}) = -1$.

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[1] The axioms **MR** and **AR** are usually stated in a stronger way as follows:

May Responsiveness (**MR**). If $p'_S < p''_S$ and $f(p'_S) \geq 0$, then $f(p''_S) = 1$. If $p'_S > p''_S$ and $f(p'_S) \leq 0$, then $f(p''_S) = -1$.

Additive responsiveness (**AR**). Let $j \notin S$. Then for any profile p_S with $f(p_S) \geq 0$, if $p_N(j) = 1$, we have $f(p_{S \cup \{j\}}) = 1$; and for any profile p_S with $f(p_S) \leq 0$, if if $p_N(j) = -1$, we have $f(p_{S \cup \{j\}}) = -1$.

However, in the presence of **N** we need not treat separately the cases when the preferences are reversed. Moreover, **N** can be proved to hold on second-order societies too, so even the arguments in section 4 below are not affected by the weak form I have chosen for the two axioms.

Then by (a) we get $p_N(j) = -1$. By **A** we obtain that $f(p_S) = f(-p_S)$; by **N** we have $f(p_S) = -f(-p_S)$, and so $f(p_S) = 0$ – contradiction. Now suppose that the proposition is proved for $|S| = n$. So let S have $n + 1$ members, and $f(p_S) = 1$ and $p_N(i_{n+1}) = 1$ and $f(p_{S-\{i_{n+1}\}}) = -1$. By induction, this happens if $p_N(i_k) \leq 0$ for all $k \leq n$ and $p_N(i_{k'}) = -1$ for some $i_{k'}$. As proved in the first step, $f(p_{\{i_{k'}, i_{n+1}\}}) = 0$. By **IIS**, $f(p_S) = 1 = f(p_{S-\{i_{k'}, i_{n+1}\}})$. But we assumed that $p_N(i_k) \leq 0$ for all $i_k \in S - \{i_{k'}, i_{n+1}\}$, and by induction (since $|S - \{i_{k'}, i_{n+1}\}| = n - 1$) we get $f(p_{S-\{i_{k'}, i_{n+1}\}}) \leq 0$ – contradiction.

Proposition 3. If a social welfare function f satisfies **WP**, **N**, **A** and **IIS**, then it satisfies **AR**.

Proof. Let $j \notin S$ and $p_N(j) = 1$. We have two cases. First, if $f(p_S) = 0$, then $f(p_{S\cup\{j\}}) = f(p_{\{j\}})$ by **IIS**. But $f(p_{\{j\}}) = p_N(j)$ by **WP** and so $f(p_{S\cup\{j\}}) = 1$, as required by **AR**. Secondly, suppose that $f(p_S) = 1$. By proposition 2b there is some $j' \in S$ such that $p_N(j') = 1$. If $f(p_{S-\{j'\}}) = 0$, then $f(p_{S\cup\{j\}}) = f(p_{(S-\{j'\})\cup\{j,j'\}}) = f(p_{\{j,j'\}})$ by **IIS**; but $f(p_{\{j,j'\}}) = 1$ by **WP** and so $f(p_{S\cup\{j\}}) = 1$. If $f(p_{S-\{j'\}}) = 1$, then we need to repeat the same procedure at most n - 1 times until by proposition 2c we get either a subsociety S' of S such that $f(p_{S'}) = 0$, or a society $S' = \{j''\}$ where $p_N(j'') = 1$, whence by **IIS** and **WP** we get that $f(p_S) = 1$, as required by **AR**.

The proof of the theorem 1c is completed once we put together proposition 3 and theorem 1b.

4. Comparing responsiveness axioms

By Theorem 1c, the axiom **IIS** characterizes μ in conjunction with three other properties: **A**, **N** and **WP**. Proposition 4 shows that the presence of **WP** is essential in this case: we cannot uniquely characterize μ if **WP** is removed.

Proposition 4. There is a social welfare function f that satisfies **A**, **N** and **IIS** and is different from μ.

Proof. Take f be the constant function: $f(p_S) = 0$ for all S.

However, **WP** is not required by the other two responsiveness axioms. They succeed to characterize μ in conjunction with only **A** and **N**. So **IIS** is weaker than both **MR** and **AR**.

On the other hand, in the presence of **A** and **N**, axioms **MR** and **AR** yield the same result. But (by Proposition 1) they are not equivalent. An explanation of this situation is that at least one of the two properties **MR** and **AR** is stronger than necessary for characterizing μ [2].

I shall argue that **AR** is weaker than **MR**. My argument appeals to second-order societies, i.e. societies that have members that are themselves societies. I show that on these societies μ still satisfies **AR**, but not **MR**. This result entails that **AR** is weaker than **MR**. Asan and Sanver (2003) and Miroiu (2004) appealed to second-order societies to axiomatize the majority rule.

A first-order society is a non-empty subset of N. In what follows, I shall also regard individuals as (degenerate) first-order societies. Second-order societies are collections of first-order societies. For example, let $\{i_1, i_2\}$ and $\{i_3, i_4\}$ be two first-order societies. They can be put together to get a new, second-order society $S^1 = \{\{i_1, i_2\}, \{i_3, i_4\}\}$. I shall denote by

[2] Obviously, an alternative explanation is that **A** and **N** are themselves too strong.

$\mathbf{S}^{[2]}$ the collection of all second-order societies. Clearly, we also need to extend the definition of a social welfare function to these new societies. Let $S = \{S_1, S_2, \ldots S_n\}$ be a second-order society. Then it is natural to put $f(p_S) = f(f(p_{S_1}), f(p_{S_2}), \ldots f(p_{S_n}))$.

The appeal to second-order societies is useful when we attempt to study two-step elections. In these cases the country is divided into jurisdictions in which local winners are elected. They are then aggregated at the 'federal' level where the final winner is elected. The most prominent example of two-step elections is provided by the Presidential elections in the USA. Roughly, the state level corresponds to first-order societies, and the federal level to a second-order society.

Proposition 5.

a) μ satisfies **AR** on $\mathbf{S}^{[2]}$.
b) μ does not satisfy **MR** on $\mathbf{S}^{[2]}$.

Proof. For part (a), let $S = \{S_1^1, \ldots S_m^1\}$ be a second-order society, where S_i^1 denotes a first-order society, and let $i \in N$. We need to show that if $\mu(p_S) \geq 0$ and $p_N(i) = 1$, then $\mu(p_{S\cup\{i\}}) = 1$. We have: $\mu(p_{S\cup\{i\}}) = \mu(\mu(p_{S_1^1}), \ldots \mu(p_{S_m^1}), p_N(i))$. By definition $\mu(p_{S\cup\{i\}}) = \text{sgn}((\sum_{i=1}^{m}\mu(p_{S_i^1}))) + 1)$. But by supposition $\text{sgn}(\sum_{i=1}^{m}\mu(p_{S_i^1})) \geq 0$, so $\sum_{i=1}^{m}\mu(p_{S_i^1}) \geq 0$ which gives $\sum_{i=1}^{m}\mu(p_{S_i^1}) + 1 > 0$. Therefore $\text{sgn}((\sum_{i=1}^{m}\mu(p_{S_i^1}))) + 1) = 1$.

For part (b), consider the second-order society $S^1 = \{\{i_1, i_2\}, \{i_3, i_4\}\}$ and a profile p_{S^1} of it given by: $p_N(i_1) = p_N(i_1) = 1$, $p_N(i_3) = p_N(i_4) = -1$. Then $\mu(p_{S^1}) = \mu(\mu(p_{\{i_1,i_2\}}), \mu(p_{\{i_3,i_4\}}))= \mu(1, -1) = 0$. Now let a profile p'_{S^1} be defined by: $p'_N(i_1) = p'_N(i_2) = 1$, $p'_N(i_3) = -1$, $p'_N(i_4) = 0$. We have $p'_{S^1} > p_{S^1}$, and **MR** entails that $\mu(p'_{S^1}) = 1$. But we can check that $\mu(p'_{S^1}) = \mu(\mu(p'_{\{i_1,i_2\}}), \mu(p'_{\{i_3,i_4\}})) = \mu(1, -1) = 0$ – contradiction[3]. So **MR** does not hold on $\mathbf{S}^{[2]}$.

To conclude, **IIS** is weaker than both **MR** and **AR**, since it needs the auxiliary property **WP** besides **A** and **N** to characterize μ. The example of the second-order societies shows that **AR** is weaker than **MR**. Given that **AR** and **MR** are independent of each other, and that by Theorem 1b the axiom **AR** in conjunction with **A** and **N** is sufficient to characterize the majority rule, it follows that the properties used by K. May (1952) to axiomatize the majority rule are too strong. This fact might explain the reluctance of many authors to appeal to **MR**.

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[3] Anonymity also fails on $\mathbf{S}^{[2]}$. To see this, take the example of the society $S^2 = \{\{i_1\}, \{i_1, i_2\}, \{i_1, i_3\}\}$. We can easily check that for each profile p_{S^2} of S^2 the preferences of the individuals i_2 and i_3 count only if the individual i_1 abstains; in all the other cases the society's preference is exactly the preference of i_1. So in this society i_1 enjoys a special status.

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