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Pair Copula Construction based Expected Shortfall estimation

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Abstract

In this note we present an algorithm for portfolio ES estimation through Pair Copula Construction. The advantages of this method are the flexibility in what dependence structure is determined, as well as the simplicity of simulation procedures. We illustrate our approach with a brief empirical application with international market indices during a crisis period, comparing with other techniques which are largely applied.

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1. Introduction

Value-at-Risk (VaR) has become the standard measure for financial market risk since it was sanctioned by the Basle Committee. However, with the huge losses suffered in last decade by many financial institutions, VaR has been often criticized. These critics have two main causes: i) VaR is not a coherent risk measure, since it does not satisfy the subadditivity condition; and ii) VaR ignores the loss (of) magnitude beyond the quantile point of interest.

As an alternative risk measure which can remedy these drawbacks, Artzner et al. (1999) introduced the Expected Shortfall (ES). In short, VaR describes the loss that can occur over a given period under a certain confidence interval, while ES is the expected loss conditional on the loss being above VaR. Since ES introduction, empirical literature has been concerned with new forms for ES estimation. Some recent methods for estimation are Cornish-Fisher approximation (Giamouridis, 2006), nonparametric econometric tools (Chen, 2008; Taylor, 2008), the use of asymmetric *t* and exponential power distributions (Zhu and Galbraith, 2011), the integral of the conditional quantile function (Leorato et al., 2012), and the weighted Nadaraya–Watson estimation (Kato, 2012).

One can note that these new methods exhibit a trend for flexible and robust estimation techniques, which can lead with the financial returns stylized facts, such as negative asymmetric leptokurtic behavior and volatility clusters. In this sense, this note has as objective **presenting ES estimation based on Pair Copula Constructions (PCC)**. A copula is a function that links univariate marginal to their multivariate distribution by splitting them, specifying joint distribution with full flexibility. Cherubini et al. (2012) is an excellent review for copula methods in finance. The great part of the research in copulas is still limited to the bivariate case. So, constructing higher dimensional copulas is the natural next step, being very possible that the most promising of these is the PCC.

Originally proposed by Joe (1996), PCC is based on a decomposition of a multivariate density into unconditional and conditional bivariate copula densities. Applications to financial data have shown that these models outperform other multivariate copula models in predicting log-returns of equity portfolios (Aas and Berg, 2009; Chollete et al., 2009; Fischer et al., 2009; Aas and Berg, 2011; Czado et al., 2012).

2. Expected Shortfall estimation through Pair Copula Constructions

Consider that financial log-returns have a marginal specification based on expectation, dispersion and random component, conform formulation (1).

$$r_{i,t} = \mu_{i,t} + \sigma_{i,t} z_{i,t}. \tag{1}$$

Where, for an asset *i* in period *t*, r_t is the log-return; μ_t is the conditional mean; σ_t is the conditional standard deviation; z_t represents the innovations white noise series, which can assume many probability distributions functions.

After isolating marginal behavior, i.e., fitting the expectation and dispersion components, it is possible to conduct a joint analysis free of this marginal influence. To that, one should transform residuals $z_{i,t}$ into pseudo-observations $\boldsymbol{u} \in [0,1]$ through the ranks as $u_i = Rank(z_i)/(length(z_i) + 1)$. This procedure is needed because copula functions domain and image definition. With these pseudo-observations one can estimate a PCC for log-returns joint specification. The two main types of PCC that have been proposed in the literature are the C (canonical)-vines and D-vines. Here we focus on the D-vine estimation, which has the density as formulation (2).

$$f(u_1, \cdots, u_n) = \prod_{k=1}^{n} f(u_k) \prod_{j=1}^{n-1} \prod_{i=i}^{n-j} c \begin{Bmatrix} F(u_i|u_{i+1}, \cdots, u_{i+j-1}), \\ F(u_{i+j}|u_{i+1}, \cdots, u_{i+j-1}) \end{Bmatrix}. \tag{2}$$

In (2), $u_1, \cdots, u_n$ are the pseudo-observations; f is the density function; $c(\cdot,\cdot)$ is a bivariate copula density and the conditional distribution functions are computed as in (3).

$$F(u_i|\boldsymbol{u}) = \frac{\partial C_{u_i,u_j|\boldsymbol{u}_{-j}}\{F(u_i|\boldsymbol{u}_{-j}),F(u_j|\boldsymbol{u}_{-j})\}}{\partial F(u_j|\boldsymbol{u}_{-j})}. \quad (3)$$

In (3), $C_{u_i,u_j|\boldsymbol{u}_{-j}}$ is the dependency structure of u_i and u_j bivariate conditional distribution conditioned on $\boldsymbol{u}_{-j}$, where the vector $\boldsymbol{u}_{-j}$is the vector $\boldsymbol{u}$ excluding the component u_j. Concerning to parameters estimation, Aas et al. (2009) propose a Maximum Likelihood (ML) estimation procedure which follows a stepwise approach.

With marginal and joint parameters already estimated, we present an algorithm for ES computation for desired out-sample period. This procedure is an extension of that proposed by Aas and Berg (2011) for VaR. This VaR calculation was successfully employed on literature in some very recent papers (see, for instance, Righi and Ceretta, 2012). Thus, the PCC based ES algorithm is as following. For each significance *p* and day *k* at out-sample period:

1. Compute forecasts of the conditional mean $\mu_{i,t+k}$ and standard deviation $\sigma_{i,t+k}$ of each asset through marginal models;
2. Simulate *N* samples $u_{i,N}$ for each asset *i* through estimated PCC;
3. Convert each set of simulation $u_{i,N}$ to $z_{i,N}$ samples through the inversion of their marginal density probability as $z_{i,N} = F_i^{-1}(u_{i,N})$;
4. For each asset *i*, determine the daily log-return *N* simulations conform marginal specification $r_{i,N,t+k} = \mu_{i,t+k} + \sigma_{i,t+k} z_{i,N}$;
5. Compute the *N* portfolio returns as $\boldsymbol{w}'\boldsymbol{r}_N$, where $\boldsymbol{w} = \{w_1, w_2, \cdots, w_n\}$ is the weights vector and $\boldsymbol{r}_N = \{r_{1,N,t+k}, r_{2,N,t+k}, \cdots, r_{n,N,t+k}\}$ is the log-returns vector of assets *1*,*2*,…,*n*;
6. Compute the VaR_{t+k}^p as the portfolio simulated returns *p*-th quantile, i.e., $VaR_{t+k}^p = \min_{\boldsymbol{w}'\boldsymbol{r}_N} \boldsymbol{w}'\boldsymbol{r}_N > G^{-1}(p)$, where *G* is the empirical distribution function of $\boldsymbol{w}'\boldsymbol{r}_N$;
7. **Compute the ES_{t+k}^p as the mean of the portfolio simulated returns below *p*-th quantile (VaR_{t+k}^p), i.e., $ES_{t+k}^p = \frac{1}{Np}\sum_{j=1}^{N} \boldsymbol{w}'\boldsymbol{r}_j * \boldsymbol{1}_{\boldsymbol{w}'\boldsymbol{r}_N < VaR_{t+k}^p}$, where *1* is the indicator function**;

3. Empirical Illustration

We used daily log-returns of the Morgan Stanley Capital International (MSCI) market indices of U.S., Germany, Brazil and Hong Kong from June 2010 to June 2012, totalizing 561 observations. These indices were chosen to avoid non-synchronism issues. The period correspond to the Eurozone crisis (Righi and Ceretta, 2011), becoming a relevant scenario for risk management tools capacity test. The last 100 observations were separated for posterior backtesting.

Initially, we estimated marginal ARMA (*m*,*n*) - GARCH (*p*,*q*) models with student's *t* innovations. Innovations distribution and lags were chosen through AIC. After, we transformed marginal residuals into pseudo-observations, and we estimated a PCC, considering the absolute Kendall's Tau as entrance order criterion. Results for these marginal and joint estimations are presented on Table 1, despite their analysis is out of scope. However, it is valid mentioning that ARMA-GARCH models presented a proper fit, as pointed by parameter significance and *Q* statistics. The same is true for PCC, all relationships presented significant parameters.

Table 1 – ARMA-GARCH e PCC estimation results for the MSCI indices log-returns at in-sample period.

Marginal Market	**EUA**	**Germany**	**Brazil**	**Hong Kong**
Mean Constant	0.0011	0.0011	0.0007	0.0008
	(0.0014)	(0.1173)	(0.3338)	(0.1509)
AR 1	-	-0.8069	-	0.6390
		(0.0000)		(0.0042)
MA 1	-	0.8259	-	-0.5782
		(0.0000)		(0.0132)
Variance Constant	0.0000	0.0000	0.0000	0.0000
	(0.1129)	(0.0000)	(0.1681)	(0.0987)
ARCH 1	0.1529	0.0855	0.0410	0.1148
	(0.0053)	(0.0635)	(0.0397)	(0.0521)
GARCH 1	0.8461	0.8980	0.9379	0.8545
	(0.0000)	(0.0000)	(0.0000)	(0.0000)
Shape	4.0781	11.5313	8.5172	5.8224
	(0.0000)	(0.0537)	(0.0063)	(0.0021)
Q (10) linear	3.9041	4.6391	10.9502	7.2061
	(0.8687)	(0.7954)	(0.2046)	(0.5145)
Q (10) squared	7.1913	8.3755	8.2166	10.9343
	(0.5161)	(0.3977)	(0.4061)	(0.2362)
Joint Relationships	**Copula**	**Parameter 1**	**Parameter 2**	
EUA | Germany	BB7	1.8674	1.0422	
		(0.0000)	(0.0000)	
Germany | Brazil	BB7	1.6351	1.1515	
		(0.0000)	(0.0000)	
Brazil | Hong Kong	BB7	1.1962	0.3572	
		(0.0000)	(0.0000)	
EUA | Brazil	Normal	0.4529	-	
		(0.0000)		
Germany | Hong Kong	Frank	1.0137	-	
		(0.0000)		
EUA | Hong Kong	Student	-0.0599	6.8181	
		(0.8720)	(0.0028)	

After estimating marginal and joint models, we performed the algorithm exposed on section 2, obtaining VaR and ES predictions for out-sample period, at 5% and 1% significance levels. We considered an equal-weighted portfolio, for question of simplicity. We backtested these predictions with tests proposed by Kupiec (1995) and Christoffersen (1998) for Var, and McNeil and Frey (2000) for ES. Further, for comparison matter, we also realized predictions using a ARMA (1,1) – GARCH (1,1) and a Dynamic Conditional Correlation (DCC) models. For ARMA – GARCH model we constructed a portfolio series as the MSCI indices mean, while for DCC we constructed mean and variance portfolio series pondering conditional means and dynamic covariance matrix by weights vector. Thus, for this two approaches, we computed VaR and ES in a parametric way, conform can be found in Christoffersen (2012), for example. Table 2 presents the backtesting results.

Table 2 – VaR and ES backtesting results for the MSCI indices log-returns at out-sample period.

Variable	PCC	GARCH	DCC
5% quantile			
Expected Violations	5	5	5
Occurred Violations	6	0	0
Kupiec test p-value	0.6559	0.0014	0.0014
Christoffersen test p-value	0.6147	0.0059	0.0059
McNeil and Frey test p-value	0.6772	-	-
1% quantile			
Expected Violations	1	1	1
Occurred Violations	1	0	0
Kupiec test p-value	1.0000	0.1563	0.1562
Christoffersen test p-value	0.9898	0.3660	0.3660
McNeil and Frey test p-value	0.4916	-	-

Results in Table 2 indicate that both VaR and ES predictions made with PCC approach were correct, neither test rejected null hypothesis. The predictions effected with GARCH and DCC models, rejected the null hypothesis for VaR tests at 5% level, and do not for 1% level. However, there was a risk super estimation, because there is not any violation. Thus, the ES backtest would have no sense. These results indicate an advantage for PCC approach, furthermore, what is reinforced considering that this out-sample is a crisis period.

4. Conclusion

In this note we presented an algorithm for portfolio ES estimation through PCC. The advantages of this method are the flexibility in what dependence structure is determined, as well as the simplicity of simulation procedures. We illustrate our approach with a brief empirical application with international market indices in a crisis period. The results of this illustration point towards more correct predictions of our approach, while other widespread techniques (GARCH and DCC) did not have the same performance.

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