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Low and High Types of Bidders in Asymmetric Auctions with A General Utility Function

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Abstract

We study asymmetric first-price auctions with n bidders. We expand the results of Fibich et al. (2002) for asymmetric first-price auctions to a general utility function. We show that for low type bidders, the equality of equilibrium bids with symmetric, uniform distribution bids holds for the general case of a utility function. For high types of bidders, those with weaker distributions bid more aggressively than stronger bidders under mild assumptions of a utility function.

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1. Introduction

Most studies of first-price auctions assume that there is symmetry in the bidders' valuations and risk neutrality, meaning that bidders' valuations are drawn by the same distribution function[1]. When we relax the symmetry assumption, it very difficult to analyze first-price auctions, and little is known about them. Furthermore, if we add complicated utility functions such as risk aversion to the mix, there is even less information available.

Maskin and Riley (2000b) proved the existence of equilibrium in asymmetric cases involving first-price auctions. There are also several works that found an explicit solution for the equilibrium function (for example, see Lebrun 1996, Maskin and Riley 2000a, Kirkegaard 2012).[2]

In our paper, we study asymmetric first-price auctions with n bidders and a general utility function. We expand the results in Fibich *et al.* (2002) in first-price auctions to the case of a general utility function. We also show that the behavior of low type bidders is identical to that of the uniform distribution in the symmetric case. For high type bidders, we assume the same utility function. Under this assumption, we prove that high type bidders with a lower density bid more aggressively than those with a higher density.

2. The Model

We consider a first-price auction with one indivisible prize. A bidder i has a private valuation for prize v_i that is drawn independently from a continuously differentiable distribution function $F_i(v)$ over the support $[0,1]$ with a strictly positive density $F_i' = f_i \geq 0$. Each bidder i submits a bid $b_i(v_i)$ independently of other bidders. Assume that there is an asymmetric, monotonic and differentiable equilibrium bid function $b_i(v_i)$. Let us define bidder i's function as $x_i = b_i(v_i)$, and his or her inverse bid function as $y_i(x_i)$. Let the $u_i(x_i, v_i) = u_i(v_i - x_i)$ be a utility function that is twice continuously differentiable and satisfies $\frac{\partial u_i(\cdot)}{\partial v_i} > 0$, $\frac{\partial u_i(\cdot)}{\partial x_i} < 0$ for all i.[3] Then, the maximization problem in first-price auctions is given by

$$\max_x V_i(x) = \left(\prod_{\substack{j=1 \\ j \neq i}}^{n} F_j(y_j(x)) u_i(v_i - x) \right) \quad i = 1, .., n$$

Thus, the solution is given by

$$\frac{\partial V_i(x)}{\partial x} = u_i(v_i - x) \sum_{\substack{j=1 \\ j \neq k}}^{n} \left(\prod_{\substack{k=1 \\ k \neq i,j}}^{n} F_k(y_k(x)) \right) f_j(y_j(x)) y_j'(x) - \prod_{\substack{j=1 \\ j \neq i}}^{n} F_j(y_j(x)) u_i'(v_i - x) = 0$$

[1] Riley and Samuelson (1981) proved the existence of equilibrium in this case of symmetry.

[2] The explicit solutions in these cases were for only two bidders and uniform distribution functions with supports or power distributions that differed from the form v^α, v^β.

[3] Maskin and Riley (2000b) proved the existence of this case.

Substituting $y_i(x_i) = v_i$ and rearranging we get

$$u_i(y_i(x) - x) \sum_{\substack{j=1 \\ j \neq i}}^{n} \frac{f_j(y_j(x))y_j'(x)}{F_j(y_j(x))} - u_i'(y_i(x) - x) = 0. \quad (1)$$

When boundary conditions are

$$y_i(0) = 0 \quad (2)$$

the lowest type bidders simply bid a zero value (see Maskin and Riley 2000b).

3. Low Type Bidders

In this section we modify the technique in Fibich *et al.* (2002) for the case of a general utility function for low type bidders. We show that in general cases, the utility function bids for low type bidders are identical to those under uniform distribution in symmetric cases.

Proposition 1 *If v is low, then*

$$b_i(v) \approx \frac{(n-1)}{n} v.$$

Proof: Taking limit (1) we get

$$\begin{aligned}
&\lim_{x \to 0} u_i(y_i(x) - x) \sum_{\substack{j=1 \\ j \neq i}}^{n} \frac{f_j(y_j(x))y_j'(x)}{F_j(y_j(x))} - u_i'(y_i(x) - x) \\
= &\sum_{\substack{j=1 \\ j \neq i}}^{n} f_j(y_j(0))y_j'(0) \lim_{x \to 0} \frac{u_i(y_i(x) - x)}{F_j(y_j(x))} - u_i'(y_i(0) - 0) \stackrel{L'Hospital}{=} \\
= &\sum_{\substack{j=1 \\ j \neq i}}^{n} f_j(y_j(0))y_j'(0) \lim_{x \to 0} \frac{u_i'(y_i(x) - x)(y_i'(x) - 1)}{f_j(y_j(x))y_j'(x)} - u_i'(y_i(0) - 0)
\end{aligned}$$

Notice that from (1) we get

$$(n-1)u_i'(0)(y_i'(0) - 1) - u_i'(0) = 0 \quad (3)$$

Rearranging (3) we get $y_i'(0) = \frac{n}{n-1}$, results in

$$b_i'(v) \approx \frac{n-1}{n}.$$

□

Proposition 1 shows us that if v is low, risk-neutral and risk-averse bidders place the same bids.

4. High Type Bidders

In this section we generalize the results of Fibich *et al.* (2002) for high type bidders. Under Assumption 1 we show that bidders with lower densities bid more aggressively than others.

Assumption 1: Let us assume that all bidders have the same utility function.

Proposition 2 *If a utility function satisfies Assumption 1, $f_i(1) > f_j(1)$ when $i \neq j$, $i, j = 1, .., n$ and v is high, then $b_i(v) < b_j(v)$.*

Proof: Rearranging (1) we get

$$\sum_{\substack{j=1 \\ j \neq i}}^{n} \frac{f_j(y_j(x))y_j'(x)}{F_j(y_j(x))} = \frac{u_i'(y_i(x) - x)}{u_i(y_i(x) - x)}.$$

According to Maskin and Riley (2000b), there is $\bar{b}$ that is the same for all bidders and satisfies for all k $F_k(y_k(\bar{b})) = 1$ and $y_i(\bar{b}) = 1$. Then for $x = \bar{b}$, we get

$$\sum_{\substack{j=1 \\ j \neq i}}^{n} f_j(1)y_j'(\bar{b}) = \frac{u_i'(1 - \bar{b})}{u_i(1 - \bar{b})}.$$

Subtracting the equation for i case from the equation for j case under Assumption 1, we get

$$f_j(1)y_j'(\bar{b}) = f_i(1)y_i'(\bar{b}) \quad i, j = 1, .., n$$

Thus, $y_j'(\bar{b}) > y_i'(\bar{b})$ namely, $b_i'(1) > b_j'(1)$ yielding the result. □

5. Discussion

As we mentioned before, asymmetric first-price auctions are very difficult to analyze given that there is no solution in a close-form of differential equations. The problem becomes even more difficult when we add general utility functions. The endpoint properties that we found in this paper can give some insights into the case of asymmetric first-price auctions. Given the difficulties of finding explicit solutions to differential equations, many studies use numerical analysis to find approximated solutions. In such cases, the endpoint properties can help the numerical analysis by gaudiness, as in Hubbard *et al.* (2012) risk-neutral case.

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