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Endogenously proportional bargaining solutions

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Abstract

This paper introduces a class of endogenously proportional bargaining solutions. These solutions are inside the class of Directional solutions, which Chun and Thomson (1987) proposed to generalize (exogenously) proportional solutions of Kalai (1977). Endogenously proportional solutions are characterized by weak Pareto optimality and continuity together with two new axioms that depend on the total payoff asymmetry of the bargaining problem. Each of these solutions satisfies the basic symmetry axiom and also a stronger axiom called total payoff symmetry.

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1. Introduction

Exogenously proportional solutions to Nash's (1950) bargaining problem are already known. The first - and most well - known member of this class is the Egalitarian solution, recommended by Rawls (1971). This solution chooses the highest utility point with equal coordinates in each bargaining problem, a subset of the n-dimensional euclidean space representing the utility alternatives available to a society with n individuals. Characterization of the Egalitarian solution was offered by Kalai (1977), who generalized this solution to a class of exogenously proportional solutions.[1] In this class, given a positive n-tuple p, the corresponding solution selects in each bargaining problem the highest utility point proportional to p.

Although the Egalitarian solution has been well studied, other exogenously proportional solutions have received less attention in the literature. Among a few studies, Roth (1979) extended Kalai's (1977) generalization to bargaining problems where utilities are not restricted to be freely disposable. Peters (1986) offered alternative characterizations of exogenously proportional solutions, focusing on 'simultaneity of issues and additivity' in bargaining games. Chun and Thomson (1987) further generalized exogenously proportional solutions to the Directional solutions, focusing on 'uncertain disagreement points' in bargaining games. Characterizations of the Directional solutions and, in particular, exogenously proportional solutions were proposed by Chun and Thomson (1987, 1990a, 1990b). Recently, Hougard and Tvede (2012) and Rachmilevitch (2012) extended proportional solutions to bargaining games with nonconvex problems.

Among all exogenously proportional solutions, Egalitarian solution is the only solution that satisfies the axiom of *symmetry*, requiring the equality of payoffs gained by all individuals in the society when the bargaining problem is symmetric. The *main question* that has motivated our study is whether there exist other bargaining solutions that have some nature of proportionality and also satisfy *symmetry*. To seek such solutions, we restrict ourselves, for notational simplicity, to 2-person bargaining problems, and we define a notion of *total payoff asymmetry* to check whether the total area of a bargaining problem is symmetric with respect to the two individuals in the society. Using this notion, we introduce an axiom called *total payoff symmetry*, requiring that the solution offers equal payoffs to both individuals when the total area of the bargaining problem is symmetric. This new axiom, which is computationally easy to check, implies *symmetry* axiom. Thus, to answer our main question, we attempt to find 'nearly' proportional solutions that satisfy *total payoff symmetry*.

The class of solutions we propose select the outcome in each bargaining problem, S, using a vector of proportionality, p, on (the weak Pareto frontier of) an *endogenously* determined area, B. This area shrinks to the line segment of points with equal coordinates only if the total area of the bargaining problem is symmetric. Since the area B, on which

[1] Kalai (1977) calls these solutions *proportional solutions*, whereas we call them *exogenously proportional solutions* to highlight the distinction between Kalai's solutions and ours.

the solution lies, depends on S, the vector of proportionality p inherited by any solution in the proposed class is endogenous to S. Thus, we call our solutions *endogenously proportional solutions*. Formally, each endogenously proportional solution associates to a given bargaining problem a vector of proportionality that is identical for any two distinct problems with the same total payoff asymmetry.

We show that the class of endogenously proportional solutions are always nonempty and independent of well-known solutions, involving the solutions by Nash (1950) and Kalai and Smorodinksy (1975), and the Equal Area solution (Anbarci and Bigelow, 1994). However, our solutions are contained by the class of Directional solutions. Our main result is that the endogenously proportional solutions are characterized by weak Pareto optimality, continuity, and two new conditions that depend on the total payoff asymmetry of a given bargaining problem. Moreover, we show that the proposed class of solutions as well as our characterization result can be trivially extended to the n-person bargaining situations.

The paper is organized as follows: Section 2 introduces the basic structures. Section 3 presents the results and gives some discussion. Finally, Section 4 concludes.

2. Basic Structures

A 0-normalized 2−person bargaining problem for a society of individuals $N = \{1, 2\}$ is denoted by S, a non-empty subset of $\mathbb{R}^2_+$, representing von Neumann-Morgenstern utilities attainable through the cooperative actions of the individuals in N.[2] If the individuals fail to agree on any point in S, then each of them receives zero utility (for notational simplicity). Hence, the bargaining problems are 0-normalized. The bargaining problem (simply, problem) S satisfies the following two conditions:

(a) S is convex and compact, and there exists $x \in S$ such that $x > 0$.[3]

(b) S is comprehensive; i.e., if $x \in S$, $y \in \mathbb{R}^2_+$, and $x \geq y$ then $y \in S$ (implying that utility is freely disposable).

Let Σ_0 denote the set of all bargaining problems. A problem S is said to be ***symmetric*** if $(y_1, y_2) \in S$ implies $(y_2, y_1) \in S$. For a problem S, a point $x \in S$ is said to be ***weakly Pareto optimal*** if there exists no $y \in S$ such that $y > x$. Let $WPO(S)$ denote the set of weakly Pareto optimal points in S. We denote the ***total payoff*** of each $X \subset \mathbb{R}^2_+$ as $TP(X) = \int_{x \in X} dx$. Note that $TP(\lambda X) = \lambda^2 TP(X)$ for all $X \subset \mathbb{R}^2_+$ and $\lambda > 0$.

For each problem S, define the sets $S_{L,\beta} = \{y \in S \,|\, \beta y^1 < y^2\}$ and $S_{R,\beta} = \{y \in S \,|\, \beta y^1 > y^2\}$ for all $\beta > 0$. For each problem S, also define $\alpha(S)$ such that $TP(S_{R,\alpha(S)}) = TP(S_{L,1})$ if $TP(S_{L,1}) < TP(S_{R,1})$, $TP(S_{L,\alpha(S)}) = TP(S_{R,1})$ if $TP(S_{L,1}) > TP(S_{R,1})$, and $\alpha(S) = 1$ if $TP(S_{L,1}) = TP(S_{R,1})$. Clearly, $\alpha(S)$ always exists and it is unique.

[2] $\mathbb{R}^2_+ = \{x \in \mathbb{R}^2 \,|\, x^i \geq 0 \text{ for all } i\}$ and $\mathbb{R}^2_{++} = \{x \in \mathbb{R}^2 \,|\, x^i > 0 \text{ for all } i\}$.

[3] Given x and y in $\mathbb{R}^2_+$, $x \geq y$ means $x^i \geq y^i$ for all i and $x > y$ means $x^i > y^i$ for all i.

We will call $\alpha(S)$ (a measure of) ***total payoff asymmetry*** of S with respect to individuals 1 and 2. A problem S is said to satisfy ***total payoff symmetry*** if $\alpha(S) = 1$.

For each problem S, define the set

$$B(S) = \begin{cases} S_{R,1} \backslash S_{R,\alpha(S)} & \text{if } \alpha(S) \in (0,1), \\ S \backslash (S_{L,1} \cup S_{R,1}) & \text{if } \alpha(S) = 1, \\ S_{L,1} \backslash S_{L,\alpha(S)} & \text{if } \alpha(S) > 1. \end{cases}$$

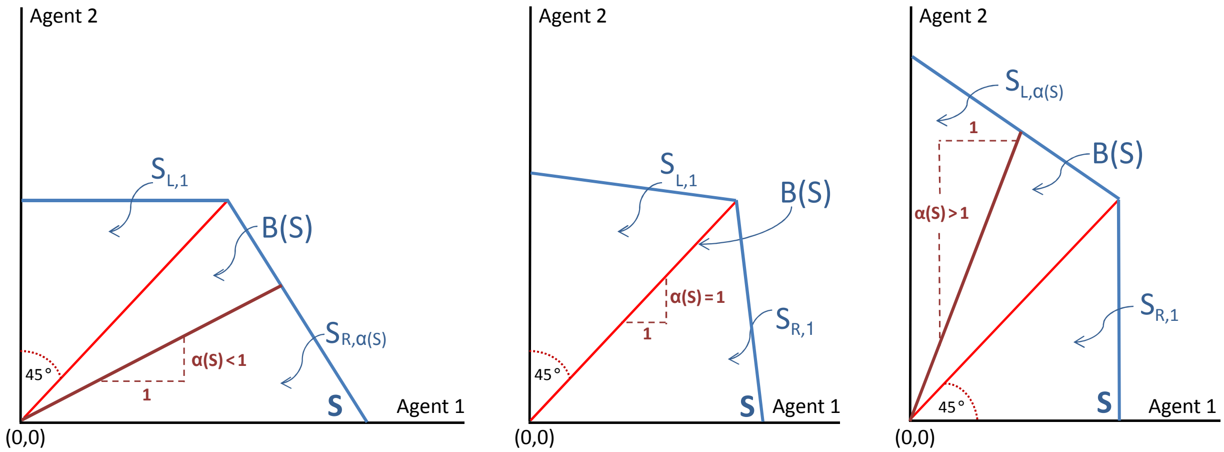

Figure 1. Basic Sets.
(a) $\alpha(S) < 1$. (b) $\alpha(S) = 1$. (c) $\alpha(S) > 1$.

Note that $B(S)$ is always nonempty.

We will call $B(S)$ the ***balancing subset*** of S, given the fact that it balances (the total payoffs of) the sets $S_{R,\alpha(S)}$ and $S_{L,1}$ if $\alpha(S) \in (0,1]$ and the sets $S_{L,\alpha(S)}$ and $S_{R,1}$ if $\alpha(S) > 1$.

Finally, a solution is a function $\mu : \Sigma_0 \to R^2_+$ such that $\mu(S) \in S$ for each $S \in \Sigma_0$.

3. Results and Discussion

We say that a solution μ is ***endogenously proportional*** if there exists a continuous function $r : \mathbb{R}_{++} \to (0,1]$ such that $\mu(S) = \lambda(S)p(S)$ for all $S \in \Sigma_0$, where $p(S) \in \mathbb{R}^2_{++}$ is such that $p^2(S)/p^1(S) = 1 - r(\alpha(S)) + r(\alpha(S))\alpha(S)$ and $\lambda(S) = \max\{t \,|\, tp(S) \in B(S)\}$. We will denote by EP the class of solutions that are endogenously proportional.

Any solution μ in the class EP is called proportional since for any pair of problems S and T with the same total payoff asymmetry (i.e., $\alpha(S) = \alpha(T)$), we have $\mu^2(S)/\mu^1(S) = \mu^2(T)/\mu^1(T)$. On the other hand, the proportionality of any solution μ in EP is endogenous since it is not invariant to changes in the problem S that affect $\alpha(S)$.

The class of solutions we propose joins the concept of proportionality with a view of fairness. According to this view, "a fair division of the gains to cooperation is one in which a particular balance of concessions is achieved" (Anbarci and Bugelow, 1994:

p.134). The class of solutions EP selects the solution outcome inside the set $B(S)$. Here, the set $B(S)$ can be viewed to be balancing the concessions of individuals 1 and 2, which are respectively $S_{R,\alpha(S)}$ and $S_{L,1}$ if $\alpha(S) \in (0,1]$ and $S_{R,1}$ and $S_{L,\alpha(S)}$ if $\alpha(S) > 1$. For instance, for any problem S with $\alpha(S) > 1$, the concession set $S_{R,1}$ of individual 1 contains from the viewpoint of him/her all favorable points in S with respect to $B(S)$ in the sense that for all $s \in S_{R,1}$ and for all $\hat{s} \in B(S)$ such that $\hat{s}^2 = s^2$, we have $s^1 > \hat{s}^1$. Similarly, for any problem S with $\alpha(S) > 1$, the concession set $S_{L,\alpha(S)}$ of individual 2 contains for him/her all favorable points in S with respect to $B(S)$ in the sense that for all $s \in S_{L,\alpha(S)}$ and for all $\hat{s} \in B(S)$ such that $\hat{s}^1 = s^1$, we have $s^2 > \hat{s}^2$. These concessions are balanced as the set $S_{L,\alpha(S)}$ is constructed against the set $S_{R,1}$ in such a way that they contain the same number of alternatives, which we measure using the notion of area (in Lebesgue measure) since the bargaining problem S is a continuum. However, the concessions of individuals 1 and 2 are in general not totally exhaustive in S. So, the set of weakly Pareto optimal points in the set $B(S)$, over which the two individuals reach a solution in EP after conceding mutually, is a continuum unless $\alpha(S) = 1$. Over this endogenously determined continuum, the bargaining outcome is solely determined by the exogenously given function r corresponding to a particular solution in EP.

Regarding the existence of endogenously proportional solutions, we have the following simple observation.

Remark 3. The class of solutions EP is non-empty.

To show that the above remark is true, we rewrite the set $B(S)$ as

$$B(S) = \{y \in S \,|\, y^2 \in Z(y^1, \alpha(S))\}$$

where

$$Z(y^1, \alpha(S)) = \begin{cases} [\alpha(S)y^1, y^1) & \text{if } \alpha(S) \in (0,1), \\ \{y^1\} & \text{if } \alpha(S) = 1, \\ (y^1, \alpha(S)y^1] & \text{if } \alpha(S) > 1. \end{cases}$$

Given the definition of endogenously proportional solutions and the fact that $B(S)$ is nonempty for all S, it is clear that EP is non-empty.

While the proportional solutions that we have described are novel to the bargaining literature, exogenously proportional solutions are already known by the work of Kalai (1977). A solution μ over Σ_0 is ***exogenously proportional*** if there exists $p \in \mathbb{R}^2_{++}$ such that $\mu(S) = \lambda(S)p$ for each $S \in \Sigma_0$, where $\lambda(S) = \max\{t \,|\, tp \in S\}$. The distinction between the class of solutions EP and Kalai's (1977) exogenously proportional solutions should be apparent, once we notice that for any exogenously determined factor of proportionality $p \in \mathbb{R}^2_{++}$ such that $p^1 \neq p^2$, we can always find a problem $S \in \Sigma_0$ such that the set $B(S)$, over which the class of solutions EP is defined, excludes the exogenous solution $\lambda(S)p$, where $\lambda(S) = \max\{t \,|\, tp \in S\}$. For instance, for any S that is symmetric, the set $B(S)$ boils down to $\{x \in S \,|\, x^1 = x^2\}$. Clearly, no exogenously proportional solution other than the Egalitarian solution can produce a solution outcome in this set. On the other hand, the Egalitarian solution is not inside the class EP either, since for any bargaining problem S with $\alpha(S) \neq 1$, the set $\{x \in S \,|\, x^1 = x^2\}$, where the Egalitarian solution selects its outcome, has an empty intersection with the set $B(S)$. (This is because the range of the function r in the definition of EP excludes the point 0, which would yield a vector of proportionality corresponding to the Egalitarian solution.)

Below, we will check whether our solutions are also independent of some well-known solutions. A solution μ is said to be

(i) the Nash (1950) solution over Σ_0 if for any $S \in \Sigma_0$

$$\mu(S) = \arg\max_{x \in S} x^1 x^2,$$

(ii) the Kalai-Smorodinsky (1975) solution over Σ_0 if for any $S \in \Sigma_0$

$$\mu(S) = \max\{x \in S \,|\, x^2/x^1 = a^2(S)/a^1(S)\}$$

where $a^i(S) = \max\{x^i \,|\, x \in S\}$ for all $i \in N$,

(iii) the Equal Area solution over Σ_0 (Anbarci and Bigelow, 1994) if for any $S \in \Sigma_0$

$$\mu(S) = \max\{x \in S \,| TP(S^L(x)) = TP(S^R(x))\}$$

where $S^L(x) = \{y \in S \,|\, y^2 \geq x^2\}$ and $S^R(x) = \{y \in S \,|\, y^1 \geq x^1\}$.

We denote the Nash, Kalai-Smorodinsky, and Equal Area solutions by $\mathcal{N}$, $\mathcal{KS}$, and $\mathcal{EA}$, respectively. We will show that none of these three solutions are endogenously proportional. Consider the problems

$$S = \text{convex hull } \{(0,0),(0,1),(1,1),(2,0)\}, \quad \text{and}$$

$$S' = \text{convex hull } \{(0,0),(0,1),(\sqrt{3},1),(\sqrt{3},0)\}.$$

Clearly, we have $\alpha^{1,2}(S) = \alpha^{1,2}(S') = 1/3$. Also, it is easy to check that $\mathcal{N}(S) = (1,1)$, $\mathcal{KS}(S) = (4/3, 2/3)$, $\mathcal{KS}(S') = (\sqrt{3}, 1)$, $\mathcal{EA}(S) = (5/4, 3/4)$, and $\mathcal{EA}(S') = (\sqrt{3}, 1)$. Obviously, the Nash solution is not endogenously proportional, since there exists no $r : \mathbb{R}_{++} \to (0,1]$ such that $\mathcal{N}^2(S)/\mathcal{N}^1(S) = 1 - r(1/3) + (1/3)r(1/3)$. We should notice that of the remaining two solutions to be checked, the Kalai-Smorodinsky solution is endogenously proportional in an obvious sense, which is different from the sense defined in our paper. On the other side, the Equal Area solution always select inside the intersection of pairwise balanced sets of a given bargaining problem, since this solution totally balances the concessions of individuals. Interestingly, a recent characterization of the Kalai-Smorodinsky solution by Anbarci (1998) shows that even the

Kalai-Smorodinsky solution can be formulated using the area notion in balancing the concessions of individuals for a particular type of bargaining problems.[4] Now, to check whether $\mathcal{KS} \in EP$, suppose that there exists a function $r : \mathbb{R}_{++} \to (0, 1]$ such that $\mathcal{KS}^2(S)/\mathcal{KS}^1(S) = 1/2 = 1-r(1/3)+(1/3)r(1/3)$, implying $r(1/3) = 3/4$. We must also have $\mathcal{KS}^2(S')/\mathcal{KS}^1(S') = 1/\sqrt{3} = 1-r(1/3)+(1/3)r(1/3)$, implying $r(1/3) = (3-\sqrt{3})/2$, a contradiction. Thus, the Kalai-Smorodinsky solution is not endogenously proportional. Finally, to check whether $\mathcal{EA} \in EP$, suppose that there exists a function $r : \mathbb{R}_{++} \to (0, 1]$ such that $\mathcal{EA}^2(S)/\mathcal{EA}^1(S) = 3/5 = 1 - r(1/3) + (1/3)r(1/3)$, implying $r(1/3) = 3/5$. We must also have $\mathcal{EA}^2(S')/\mathcal{EA}^1(S') = 1/\sqrt{3} = 1 - r(1/3) + (1/3)r(1/3)$, implying $r(1/3) = (3 - \sqrt{3})/2$, a contradiction. Therefore, the Equal Area solution is not endogenously proportional.

We can finally check that our solutions lie inside the class of Directional solutions, to which Chun and Thomson (1990) further generalized exogenously proportional solutions. To see this, consider a class of 2-person problems Σ, where each problem involves a bargaining set $S \subset \mathbb{R}^2$ satisfying the usual feasibility assumptions and a disagreement point d in S, where the individuals end up if they fail to agree on a point in S. If for a given solution μ there exists a continuous function p from the set of feasible bargaining sets to the 2-dimensional simplex Δ^2 such that for all $(S, d) \in \Sigma$, $\mu(S, d) = d + \lambda(S)p(S)$, where $\lambda(S) = \max\{t \,|\, d+tp(S) \in S\}$, then μ is called ***the Directional solution*** relative to p.

To make a comparison with the Directional solutions, we will extend the class of solutions EP from Σ_0 to Σ. Note that to each problem $(S, d) \in \Sigma$ corresponds a unique problem in Σ_0: the individually rational set of points in S with respect to d, i.e., $IR(S, d) = \{x \in S - d \,|\, x \geq 0\}$. So, given any $\mu \in EP$, we can define a solution on Σ, $\varphi \equiv \varphi(\mu)$, by $\varphi(S, d) = d + \mu(IR(S, d))$. Clearly, the solution φ is a Directional solution since the solution μ is endogenously proportional.[5] Thus, one can safely claim that each endogenously proportional solution is *essentially* a Directional solution, with the nonessential difference arising from the normalization $d \equiv 0$ in our model.

The above observation leads us to note an important advantage of endogenously proportional solutions as a subfamily of Directional solutions over the Kalai-Smorodinsky solution, as already addressed in the work of Chun and Thomson (1987). This work characterizes Directional solutions, and a subfamily of them, namely exogenously proportional

[4] A new axiom, called *Balanced Focal Point*, introduced by Anbarci (1998) for an alternative characterization of the Kalai-Smorodinsky solution over the domain Σ_0 requires that if $S \in \Sigma_0$ and $S = \text{convex hull}\{(0, 0), (0, \lambda b), (a, b), (\lambda a, 0)\}$, where $\lambda \in [1, 2]$, then $\mu(S) = (a, b)$. For each such S and for each $\lambda \in (1, 2]$, the sets $S^1(\lambda) = \text{convex hull}\{(0, 0), (a, b), (\lambda a, 0)\}$ and $S^2(\lambda) = \text{convex hull}\{(0, 0), (a, b), (0, \lambda b)\}$ can be considered to be the concessions individuals 1 and 2 respectively need to make in order to achieve $\mu(S) = (a, b)$. These concessions are balanced in the Lebesgue measure, as $TP(S^1(\lambda)) = TP(S^2(\lambda)) = (\lambda - 1)ab/2$.

[5] To see that this result directly follows from the definitions of the two classes of solutions, we should note that without loss of generality we can restrict $p(S)$ to lie in the 2-dimensional simplex Δ^2 as in the definition of the Directional solutions.

solutions (or weighted Egalitarian solutions), using a new axiom called *Disagreement Point Concavity*. According to this axiom, if there is uncertainty in the disagreement point $\tilde{d}$ such that it may take two distinct values d_1 and d_2 under some probability distribution function and if this uncertainty will be resolved tomorrow, then solving the bargaining problem today taking the disagreement point as the expected value of $\tilde{d}$ under the given probability distribution function will be preferred by all individuals to waiting until tomorrow and solving the bargaining problem with the realized value of $\tilde{d}$.[6] As already addressed by Chun and Thomson (1987), the Kalai-Smorodinksy solution, which is endogenously proportional in an alternative sense, does not satisfy the axiom of *Disagreement Point Concavity*, unlike the Directional solutions, which contain endogenously proportional solutions we propose.

Below, we present four axioms to characterize our solutions.

Weak Pareto Optimality (WPO): $\mu(S) \in WPO(S)$.

Continuity (CON): If $\{S_k\}$ converges in the Hausdorff topology to S, then $\{\mu(S_k)\}$ converges to $\mu(S)$.

Balancedness (BAL): $\mu(S) \in B(S)$.

Invariance of Payoffs under Constant Total Payoff Asymmetry (IP): If S, T are such that $\alpha(S) = \alpha(T)$, then $\mu^2(S)/\mu^1(S) = \mu^2(T)/\mu^1(T)$.

The first two axioms are well known. The third axiom, *BAL*, requires that the vector of proportionality corresponding to any solution depends on the balancing subset of the bargaining problem. Finally, *IP* requires that if the total payoff asymmetry is the same in two distinct problems, then the utility of individual 2 relative to individual 1 must also be the same in these problems. Below, we will show that *WPO* and *IP* together imply the well known *homogeneity* axiom. We will use this result in proving our characterization theorem.

Homogeneity (HOM). $\mu(cS) = c\mu(S)$ for all $c > 0$.

Lemma 1. *A solution satisfies HOM if it satisfies WPO and IP.*

Proof. Let a solution μ satisfy *WPO* and *IP*. Pick any S and $c > 0$. By *WPO*, $\mu(S) \in WPO(S)$ and $\mu(cS) \in WPO(cS)$. It follows that $c\mu(S) \in WPO(cS)$ since $cWPO(S) = WPO(cS)$. Clearly, $\alpha(cS) = \alpha(S)$. Then, by *IP*, $\mu^2(cS)/\mu^1(cS) = \mu^2(S)/\mu^1(S)$. Suppose $\mu^1(cS) > c\mu^1(S)$; then $\mu(cS) > c\mu(S)$, contradicting $c\mu(S) \in WPO(cS)$. On the other

[6] For our domain Σ, this axiom by Chun and Thomson (1987) is formally stated as follows: For all $(S_1, d_1), (S_2, d_2) \in \Sigma$ and for all $\alpha \in [0, 1]$, if $S_1 = S_2 \equiv S$, then $F(S, \alpha d_1 + (1-\alpha)d_2) \geq \alpha F(S, d_1) + (1-\alpha)F(S, d_2)$.

hand, $\mu^1(cS) < c\mu^1(S)$ would imply $\mu(cS) < c\mu(S)$, contradicting $\mu(cS) \in WPO(cS)$. So, we must have $\mu^1(cS) = c\mu^1(S)$, implying $\mu(cS) = c\mu(S)$. □

Theorem 1. *A solution on Σ_0 satisfies WPO, CON, BAL, and IP if and only if it is endogenously proportional.*

Proof. Obviously, any solution in the class EP satisfies all four axioms. Conversely, let μ be a solution satisfying *WPO, CON, BAL,* and *IP*. Pick $\phi \in \mathbb{R}_{++}$. Consider the problem $D(\phi) = \{y \in \mathbb{R}^2_+ \,|\, y^1 \leq 1 \text{ and } y^2 \leq \sqrt{\phi}\}$. We have $\alpha(D(\phi)) = \phi$, since

$$TP(D_{L,1}(\phi)) = \frac{\phi}{2} = TP(D_{R,\phi}(\phi)) \quad \text{if} \;\; \phi \in (0,1], \quad \text{and}$$
$$TP(D_{L,\phi}(\phi)) = \frac{1}{2} = TP(D_{R,1}(\phi)) \quad \text{if} \;\; \phi > 1.$$

By BAL, we have $\mu(D(\phi)) \in B(D(\phi))$, implying

$$\frac{\mu^2(D(\phi))}{\mu^1(D(\phi))} \in \begin{cases} [\phi, 1) & \text{if } \;\phi \in (0,1), \\ \{1\} & \text{if } \;\phi = 1, \\ (1, \phi] & \text{if } \;\phi > 1. \end{cases}$$

Let

$$r(\phi) = \frac{1}{\phi - 1}\left(\frac{\mu^2(D(\phi))}{\mu^1(D(\phi))} - 1\right)$$

if $\phi \neq 1$. Clearly, $r(\phi) \in (0,1]$ for all $\phi \in \mathbb{R}_{++}\backslash\{1\}$. Let $r(1) = \lim_{\phi\to1} r(\phi)$. (Note that $\mu^2(D(\phi))/\mu^1(D(\phi))$ is continuous in ϕ, since μ satisfies CON; hence the above limit exists.) Thus, we have constructed a continuous function $r : \mathbb{R}_{++} \to (0,1]$.

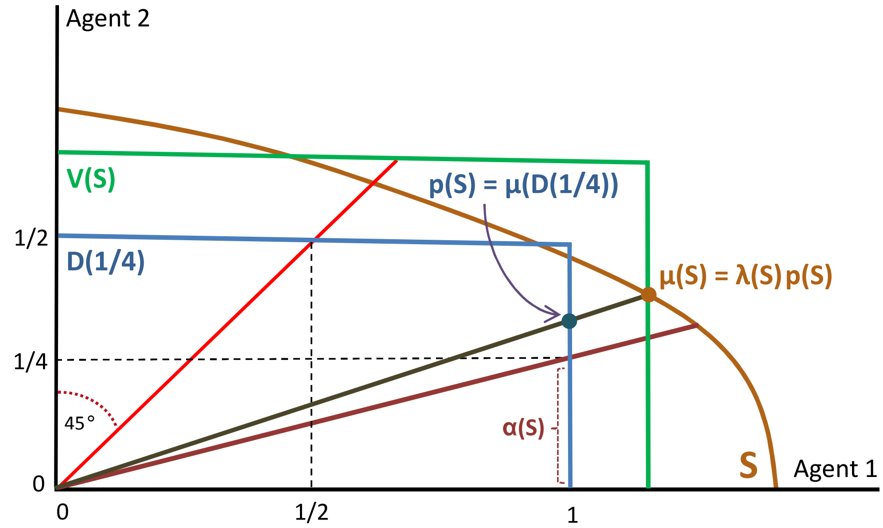

Figure 2. Sketch of the Proof for $\alpha(S) = 1/4$.

Now pick a problem S. Let $p(S) = \mu(D(\alpha(S))$. By construction, $p^2(S)/p^1(S) = 1 - r(\alpha(S)) + r(\alpha(S))\alpha(S)$. Let $\lambda(S) = \max\{t \mid tp(S) \in B(S)\}$. Clearly, $\lambda(S)p(S) \in WPO(S)$ since μ satisfies *WPO*. Consider the problem $V(S) = \lambda(S)D(\alpha(S))$. If $\alpha(S) \in (0, 1]$,

$$\begin{aligned} TP(V_{L,1}(S)) &= [\lambda(S)]^2 TP(D_{L,1}(\alpha(S))) = [\lambda(S)]^2 TP(D_{R,\alpha(S)}(\alpha(S))) \\ &= TP(V_{R,\alpha(S)}(S)). \end{aligned}$$

On the other hand, if $\alpha(S) > 1$,

$$\begin{aligned} TP(V_{R,1}(S)) &= [\lambda(S)]^2 TP(D_{R,1}(\alpha(S))) = [\lambda(S)]^2 TP(D_{L,\alpha(S)}(\alpha(S))) \\ &= TP(V_{L,\alpha(S)}(S)). \end{aligned}$$

Thus, $\alpha(V(S)) = \alpha(D(\alpha(S))) = \alpha(S)$. Also, $\mu(V(S)) = \lambda(S)\mu(D(\alpha(S))) = \lambda(S)p(S)$, since μ satisfies *HOM* by Lemma 1. By *IP*, $\mu^2(S)/\mu^1(S) = \mu^2(V(S))/\mu^1(V(S))$, since $\alpha(V(S)) = \alpha(S)$. Moreover, $\mu(V(S)) \in WPO(V(S))$, since μ satisfies *WPO*. Then, $\mu^1(V(S)) = \mu^1(S)$, for otherwise we would have either $\mu(V(S)) > \mu(S)$ contradicting $\mu(S) \in WPO(S)$ or $\mu(V(S)) < \mu(S)$ contradicting $\mu(V(S)) = \lambda(S)p(S) \in WPO(S)$. Therefore, $\mu(S) = \mu(V(S)) = \lambda(S)p(S)$. □

The axioms *WPO* and *CON* are also satisfied by exogenously proportional solutions, as already shown by Kalai (1977). Besides, these solutions satisfy *IP* as well, since by definition the vector of proportionality of any exogenously proportional solution is invariant to changes in the bargaining problem. Thus, endogenously and exogenously proportional solutions are only distinguished, in our characterization, by the *balancedness* axiom. It should be evident from the definition of endogenously proportional solutions that any possible alternative characterization of the class *EP* may constantly depend on the axiom *BAL*. This dependence is similar to the appearance of the *strong individual rationality* (*SIR*) axiom in three alternative characterizations of exogenously proportional solutions offered by Kalai (1977).[7] The axiom *SIR* requires that for each problem the solution should assign a positive utility to each individual.[8] The need for *SIR* by exogenously proportional solutions is obvious as these solutions restrict the vector of proportionality to strictly positive pairs of real numbers. On the other hand, *SIR* is not strong enough to account for the demanding restrictions our solutions put on the vector of proportionality corresponding to each problem. The restrictions put by any solution in *EP* require, for each problem, the exact knowledge of the total payoff asymmetry, hence the direct reflection of these restrictions onto an axiom like *BAL* seems to be inevitable.

[7]Kalai (1977) shows that exogenously proportional solutions are characterized by *WPO*, *HOM*, *SIR* together with *monotonicity* or *step-by-step negotiations* or a collection of three axioms involving *independence of irrelevant alternatives*, *individual monotonicity* and *continuity*.

[8]Note that *WPO* and *BAL* together imply *SIR*. Therefore any solution in the class *EP* trivially satisfies *SIR*.

The separation of endogenously and exogenously proportional solutions with regard to the *balancedness* axiom implies that the two classes of solutions also differ with respect to their relation to a basic axiom in the bargaining literature, called *symmetry*.

Symmetry. If S is symmetric, then $\mu^1(S) = \mu^2(S)$.

While *symmetry* is satisfied by no exogenously proportional solution except for the Egalitarian solution, it is satisfied by every endogenously proportional solution. The reason is that this axiom is implied by BAL, because if S is symmetric, then $\alpha(S) = 1$ and $B(S) = \{y \in S \,|\, y_1 = y_2\}$. In fact, endogenously proportional solutions satisfy a stronger form of symmetry, as well.

Total Payoff Symmetry. If S is total payoff symmetric, then $\mu^1(S) = \mu^2(S)$.

Total payoff symmetry implies *symmetry*, since every bargaining problem is total payoff symmetric if it is symmetric. But, the converse is not true. To see this, consider $S =$ convex hull $\{(0,0),(0,3/2),(1/2,3/2),(1,1),(7/4,0)\}$. Clearly, S is not symmetric, but it is total payoff symmetric since $TP(S_{L,1}) = TP(S_{R,1}) = 7/8$ and $\alpha(S) = 1$.

When we eliminate the axiom *IP* from our list of characterizing axioms, we can further generalize our solutions to a family that we call *total payoff balancing class of solutions*. Obviously, each bargaining solution satisfying *WPO* and *CON* can be extended to be a member of this general class of solutions, restricting the solution outcome on each problem S to the set $B(S)$.

Finally, we will show that our solutions can be extended to the n-person case as follows: Consider a society of individuals $N = \{1, 2, \ldots, n\}$. Let Σ_0^n denote the n-person extension of the set of 2-person bargaining problems Σ_0. For each problem $S \in \Sigma_0^n$ and distinct individuals i and j, define the sets $S^{i,j}_{L,\beta} = \{y \in S \,|\, \beta y^i < y^j\}$ and $S^{i,j}_{R,\beta} = \{y \in S \,|\, \beta y^i > y^j\}$ for each $\beta > 0$. For each problem $S \in \Sigma_0^n$ and distinct individuals i and j, also define $\alpha^{i,j}(S)$ such that $TP(S^{i,j}_{R,\alpha^{i,j}(S)}) = TP(S^{i,j}_{L,1})$ if $TP(S^{i,j}_{L,1}) < TP(S^{i,j}_{R,1})$, $TP(S^{i,j}_{L,\alpha^{i,j}(S)}) = TP(S^{i,j}_{R,1})$ if $TP(S^{i,j}_{L,1}) > TP(S^{i,j}_{R,1})$, and $\alpha^{i,j}(S) = 1$ if $TP(S^{i,j}_{L,1}) = TP(S^{i,j}_{R,1})$. Clearly, $\alpha^{i,j}(S)$ always exists and it is unique. Moreover, since $S^{i,j}_{L,\beta} = S^{j,i}_{R,1/\beta}$ and $S^{i,j}_{R,\beta} = S^{j,i}_{L,1/\beta}$, we have $\alpha^{i,j}(S) = 1/\alpha^{j,i}(S)$ for all $i, j \in N$.

We will call $\alpha^{i,j}(S)$ (a measure of) *pairwise total payoff asymmetry* of S with respect to individuals i and j. A problem S is said to satisfy *pairwise total payoff symmetry* with respect to individuals i and j if $\alpha^{i,j}(S) = 1$. Furthermore, S is said to satisfy total payoff symmetry if it satisfies pairwise total payoff symmetry with respect to each pair of individuals in the society.

For each problem S and distinct individuals i and j, define the following set, called the *pairwise balancing subset* of S with respect to individuals i and j:

$$B^{i,j}(S) = \begin{cases} S^{i,j}_{R,1} \backslash S^{i,j}_{R,\alpha^{i,j}(S)} & \text{if } \alpha^{i,j}(S) \in (0,1), \\ S \backslash (S^{i,j}_{L,1} \cup S^{i,j}_{R,1}) & \text{if } \alpha^{i,j}(S) = 1, \\ S^{i,j}_{L,1} \backslash S^{i,j}_{L,\alpha^{i,j}(S)} & \text{if } \alpha^{i,j}(S) > 1. \end{cases}$$

Note that $B^{i,j}(S)$ is always nonempty. Moreover, one can easily check that $B^{i,j}(S) = B^{j,i}(S)$.

Now, for each problem S and individual i, define $\boldsymbol{\alpha}^i(S) = (\alpha^{i,j}(S))_{j \neq i}$. We say that over the domain Σ_0^n, a solution μ is ***endogenously proportional relative to individual*** i if there exists a continuous function $r^{i,j} : \mathbb{R}_{++}^{n-1} \to (0,1]$ for all $j \neq i$ such that $\mu(S) = \lambda(S)p(S)$ for all $S \in \Sigma_0^n$, where $p(S) \in \mathbb{R}_{++}^n$ is such that $p^j(S)/p^i(S) = 1 - r^{i,j}(\boldsymbol{\alpha}^i(S)) + r^{i,j}(\boldsymbol{\alpha}^i(S))\alpha^{i,j}(S)$ for all $j \neq i$ and $\lambda(S) = \max\{t \,|\, tp(S) \in \cap_{j \neq i} B^{i,j}(S)\}$.

We will denote by EP_i the class of solutions that are endogenously proportional relative to individual i. Clearly, the class EP_i is nonempty for all i. One can naturally ask whether $\cap_{i=1}^n EP_i$ is non-empty, too. Unfortunately, we do not know the answer when $n > 2$ and leave it to future work.[9] When $n = 2$, we can easily show that the answer is 'yes'. Indeed, we have $EP_1(S) = EP_2(S)$ for all $S \in \Sigma_0^2$. (This is why at the beginning of this section we have defined endogenously proportional solutions in the 2-person case only relative to individual 1.) To see this, pick any $\mu \in EP_1$ and let $r^{1,2} : \mathbb{R}_{++} \to (0,1]$ be the function that generates μ. Pick any $S \in \Sigma_0^n$. We have $\boldsymbol{\alpha}^1(S) = \alpha^{1,2}(S)$, thus $r^{1,2}(\boldsymbol{\alpha}^1(S)) = r^{1,2}(\alpha^{1,2}(S))$ and $\mu^2(S)/\mu^1(S) = 1 - r^{1,2}(\alpha^{1,2}(S)) + r^{1,2}(\alpha^{1,2}(S))\alpha^{1,2}(S)$. Using the fact $\boldsymbol{\alpha}^2(S) = \alpha^{2,1}(S) = 1/\alpha^{1,2}(S)$, it is easy to check that $\mu^1(S)/\mu^2(S) = 1 - r^{2,1}(\alpha^{2,1}(S)) + r^{2,1}(\alpha^{2,1}(S))\alpha^{2,1}(S)$ if

$$r^{2,1}(\alpha^{2,1}(S)) = \frac{r^{1,2}(\alpha^{1,2}(S))\alpha^{1,2}(S)}{1 - r^{1,2}(\alpha^{1,2}(S)) + r^{1,2}(\alpha^{1,2}(S))\alpha^{1,2}(S)}.$$

Since $r^{2,1}(\alpha^{2,1}(S)) \in (0,1]$, we have $\mu \in EP_2$.

Now, consider the following extensions of the axioms *BAL* and *IP* to the n-person case.

Balancedness Relative to Individual i (BAL-i): $\mu(S) \in \cap_{j \neq i} B^{i,j}(S)$.

Invariance of Payoffs Relative to Individual i under Constant Pairwise Total Payoff Asymmetry (IP-i): If S, T are such that $\alpha^{i,j}(S) = \alpha^{i,j}(T)$ for some i and $j \neq i$, then $\mu^j(S)/\mu^i(S) = \mu^j(T)/\mu^i(T)$.

One can easily check that a solution on Σ_0^n satisfies *WPO, CON, BAL*$-i$, and *IP*$-i$ if and only if it is in EP_i.

4. Conclusions

In this paper, we have introduced a class of endogenously proportional solutions. These solutions are characterized by *weak Pareto optimality*, *continuity* and two new axioms that depend on the total payoff asymmetry of a given problem.

[9] The future work may also study an interesting conjecture proposed by an anonymous reviewer of this paper stating that if $\cap_{i \in I} EP_i \neq \emptyset$ for every subset $I \subset \{1, 2, \ldots, n\}$ that satisfies $|I| = n - 1$, then $\cap_{i=1}^n EP_i \neq \emptyset$.

Endogenously proportional solutions satisfy a stronger form of the *symmetry* axiom, while exogenously proportional solutions, except for the Egalitarian solution, fail to satisfy *symmetry*. Definitely, for non-Egalitarian members of the Kalai's (1977) class of solutions this is not a deficiency per se, since in environments where the players may not have the same bargaining power, asking for *symmetry* would be unreasonable. On the other hand, in environments where the bargaining problem is known to intrinsically contain the bargaining power of the players, it would be natural to focus on solutions that choose symmetric outcomes in symmetric problems. The solutions we propose may enable players in such environments to use proportional solutions without dispensing with *symmetry*. However, one difficult problem that was already addressed by Kalai (1977) for exogenously proportional solutions is what the vector of proportionality should be. For the case of each endogenously proportional solution, this problem boils down to how the weight function r, which determines the direction of the solution inside the set $B(S)$ for any problem S, should be constructed.

Endogenously proportional solutions, like the Equal Area solution, depend on the area (in Lebesgue measure) of each element inside the solution-relevant partition of a bargaining problem S. Naturally, the area-dependent bargaining solutions have the advantage of depending on the geometry of the whole of S, and thus they are sensitive to every possible change, however small, in the feasible alternatives over which individuals will bargain. But, since the area of any subset of S can only be calculated using the products of von Neumann-Morgenstern utility numbers of individuals, the area-dependent solutions are subject to the same criticism as was directed towards the Nash solution by Rubinstein, Srafa and Thomson (1992, p. 1972) on the basis that "the meaning of a product of two von Neumann-Morgenstern utility is not clear". As a remedy to the problem in the case of the Nash solution, Rubinstein, Srafa and Thomson (1992) replaced the geometric bargaining setup in the Nash's original bargaining problem, which was endowed with cardinal utility numbers, with a setup involving only alternatives and ordinal preferences. The new language in this alternative setup not only provided a straightforward interpretation of the ordinal extension of the Nash solution as well as the ordinal translations of the Nash's characterizing axioms but also allowed the extension of the Nash solution to a class of non-expected utility preferences. Using this ordinal framework, Anbarci and Bigelow (1994) showed that the Equal Area solution can be extended to domains where preference orderings cannot be represented by von Neumann-Morgenstern utility functions. Future research can use this new framework to generalize endogenously proportional solutions, as well.

Finally, we believe that new bargaining solutions can be derived from the already known solutions in the literature, restricting the outcome chosen by any proposed solution to lie in the intersection of the pairwise balancing subsets of the bargaining problem relative to a given individual. This procedure can be especially useful for finding symmetric extensions of solutions that fail to satisfy *symmetry*.

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