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On the nucleolus of 2 x 2 assignment games

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Abstract

We provide explicit formulas for the nucleolus of an arbitrary assignment game with two buyers and two sellers. Five different cases are analyzed depending on the entries of the assignment matrix. We extend the results to the case of 2 x m or m x 2 assignment games.

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1 Introduction

Assignment games were introduced by Shapley and Shubik (1972). They represent two-sided markets, whose agents are, let us say, buyers and sellers. When a member of one side is paired with a member of the other side, a non-negative real number is associated. It is the potential profit of that pairing. This setup is represented by a matrix, namely the assignment matrix. The worth of a coalition is defined as the maximum profit obtained by matching buyers to sellers within the coalition.

Given an optimal matching, the main question at issue focuses on how agents share the profit of the whole market. The core is the main solution and it is defined as those allocations of the worth of the grand coalition such that no subcoalition can improve upon. Shapley and Shubik prove that the core of an assignment game is always a nonempty polyhedron which coincides with the set of solutions of the dual linear program associated with the (linear sum) optimal assignment problem. It can be described in terms of the assignment matrix entries, that is, there is no need to compute the characteristic function. Usually the core contains many points and it becomes necessary to select one of them. One of the most outstanding core selections is the nucleolus (Schmeidler, 1969).

The nucleolus corresponds with the unique core element that lexicographically minimizes the vector of non-increasingly ordered excesses of coalitions. For assignment games, only excesses of individual and mixed-pair coalitions matter. Solymosi and Raghavan (1994) use this fact to provide a specific algorithm that computes the nucleolus of an arbitrary square assignment game, but there is no explicit formula, even for assignment games with few agents.

Recently, Llerena and Núñez (2011) have characterized the nucleolus of a square assignment game from a geometric point of view. Making use of this geometric characterization we give specific formulae of the nucleolus for 2×2 assignment games. The paper is addressed to provide such formulae and extend the above method to $2 \times m$ or $m \times 2$ assignment games. We expect that this work be a basis to analyze the general case.

2 Preliminaries

An assignment market (M, M', A) is defined by two disjoint finite sets: M, the set of buyers, and M', the set of sellers, and a non-negative matrix $A = (a_{ij})_{i\in M, j\in M'}$ which represents the profit obtained by each mixed-pair $(i, j) \in M \times M'$. To distinguish the j-th seller from the j-th buyer we will write the former as j' when needed.

A *matching* $\mu \subseteq M \times M'$ between M and M' is a bijection from $M_0 \subseteq M$ to $M_0' \subseteq M'$ such that $|M_0| = |M_0'| = \min\{|M|, |M'|\}$. We write $(i, j) \in \mu$ as well as $j = \mu(i)$ or $i = \mu^{-1}(j)$. If for some buyer $i \in M$ there is no $j \in M'$ such that $(i, j) \in \mu$ we say that i is unmatched by μ and similarly for sellers. The set of all matchings from M to M' is represented by $\mathcal{M}(M, M')$. A matching $\mu \in \mathcal{M}(M, M')$ is *optimal* for (M, M', A) if $\sum_{(i,j)\in\mu} a_{ij} \geq \sum_{(i,j)\in\mu'} a_{ij}$ for any $\mu' \in \mathcal{M}(M, M')$. We denote by $\mathcal{M}_A^*(M, M')$ the set of all optimal matchings. Shapley and Shubik (1972) associate to any assignment market a game in coalitional form $(M \cup M', w_A)$ called the *assignment game* where the worth of a coalition formed by $S \subseteq M$ and $T \subseteq M'$ is $w_A(S \cup T) = \max\limits_{\mu\in\mathcal{M}(S,T)} \sum_{(i,j)\in\mu} a_{ij}$, and any coalition formed by buyers or sellers gets zero.

The *core* of the assignment game, $C(w_A)$, is defined as those allocations $(u; v) \in \mathbb{R}^M \times \mathbb{R}^{M'}$ satisfying $u(M) + v(M') = w_A(M \cup M')$ and $u(S) + v(T) \geq w_A(S \cup T)$ for all $S \subseteq M$ and $T \subseteq M'$ where $u(S) = \sum_{i\in S} u_i$, $v(T) = \sum_{j\in T} v_j$, $u(\emptyset) = 0$ and $v(\emptyset) = 0$. It is always non-empty.

Given an optimal matching $\mu \in \mathcal{M}_A^*(M, M')$, the core of the assignment game can be easily described as the set of non-negative payoff vectors $(u; v) \in \mathbb{R}_+^m \times \mathbb{R}_+^{m'}$ such that

$$u_i + v_j \geq a_{ij} \text{ for all } i \in M,\ j \in M', \tag{1}$$

$$u_i + v_j = a_{ij} \text{ for all } (i, j) \in \mu, \tag{2}$$

and all agents unmatched by μ get a null payoff.

Now we define the nucleolus of an assignment game, taking into account that its core is always non-empty. Given an allocation in the core, $x \in C(w_A)$, define for each coalition S its excess as $e(S, x) := v(S) - \sum_{i\in S} x_i$. As it is known (see Solymosi and Raghavan, 1994) that the only coalitions that matter are the individual and mixed-pair ones, define the vector $\theta(x)$ of excesses of individual and mixed-pair coalitions arranged in a non-increasing order.

Then the *nucleolus* of the game $(M \cup M', w_A)$ is the unique alllocation $\nu(w_A) \in C(w_A)$ which minimizes $\theta(x)$ with respect to the lexicographic order over the set of core allocations.

Llerena and Núñez (2011) characterize the nucleolus of a square assignment game from a geometric point of view. The nucleolus is the unique allocation that is the midpoint of some well-defined segments inside the core. To be precise we define the maximum transfer from a coalition to their optimally assigned partners. Given any assignment market (M, M', A), an optimal matching $\mu \in \mathcal{M}_A^*(M, M')$ and an arbitrary coalition $\emptyset \neq S \subseteq M$, we define

$$\delta^A_{S,\mu(S)}(u;v) := \min_{i \in S,\ j \in M' \setminus \mu(S)} \{u_{i,} \quad u_i + v_j - a_{ij}\} \tag{3}$$

$$\delta^A_{\mu(S),S}(u;v) := \min_{j \in \mu(S), i \in M \setminus S} \{v_{j,} \quad u_i + v_j - a_{ij}\} \tag{4}$$

for any core allocation $(u;v) \in C(w_A)$.

It is easy to see that expression (3) represents the largest amount that can be transferred from players in S to players in $\mu(S)$ with respect to the core allocation $(u;v)$ while remaining in the core, that is,

$$\delta^A_{S,\mu(S)}(u;v) = \max\left\{\varepsilon \geq 0 \mid \left(u - \varepsilon 1_S; v + \varepsilon 1_{\mu(S)}\right) \in C(w_A)\right\},$$

where 1_S and $1_{\mu(S)}$ represent the characteristic vectors[1] associated with coalition $S \subseteq M$ and $\mu(S) \subseteq M'$, respectively.

Llerena and Núñez (2011) prove that the nucleolus of a square assignment market is characterized as the unique core allocation $(u;v) \in C(w_A)$ where $\delta^A_{S,\mu(S)}(u;v) = \delta^A_{\mu(S),S}(u;v)$ for any $\emptyset \neq S \subseteq M$ and $\mu \in \mathcal{M}_A^*(M, M')$, what we name the bisection property.

3 Main results

Consider the 2×2 assignment market represented by a matrix

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix},$$

[1] Given $S \subseteq N = \{1, \ldots, n\}$, $1_S \in \mathbb{R}^n$ is such that $1_{S,i} = 1$, if $i \in S$, and zero otherwise.

and denote by M_2^+ the set of all these matrices with non-negative entries. From now on and without loss of generality, we assume that the following normalization conditions hold:

$$\begin{aligned} a_{11}+a_{22} &\geq a_{12}+a_{21}, \\ a_{11}\geq a_{22}, &\quad a_{12}\geq a_{21}. \end{aligned} \tag{5}$$

These conditions mean that the main diagonal is optimal and is sorted from highest to lowest. Sectors are interchangeable so that the entries outside the main diagonal are ordered.

Our main results (Theorem 3.1 and 3.2) provide the expression of the nucleolus of an arbitrary 2×2 assignment game. We show that there are five different cases, depending on some linear inequalities concerning the matrix entries.

Their proofs rely on the characterization of the nucleolus given by Llerena and Núñez (2011). Recall that if (M, M', A) is a square assignment market $|M| = |M'| = 2$, a core allocation $(u; v) = (u_1, u_2; v_1, v_2) \in C(w_A)$ is the nucleolus of $(M \cup M', w_A)$ if and only if

$$\begin{aligned} \delta_{\{1\},\{1'\}}(u;v) &= \delta_{\{1'\},\{1\}}(u;v), \\ \delta_{\{2\},\{2'\}}(u;v) &= \delta_{\{2'\},\{2\}}(u;v), \quad \text{and} \\ \delta_{\{1,2\},\{1',2'\}}(u;v) &= \delta_{\{1',2'\},\{1,2\}}(u;v). \end{aligned}$$

Equivalently, using (3) and (4) the above conditions can be written as:

$$\min\{u_1, u_1+v_2-a_{12}\} = \min\{v_1, v_1+u_2-a_{21}\} \geq 0, \tag{6}$$

$$\min\{u_2, u_2+v_1-a_{21}\} = \min\{v_2, v_2+u_1-a_{12}\} \geq 0, \tag{7}$$

$$\min\{u_1, u_2\} = \min\{v_1, v_2\}. \tag{8}$$

being $v_i = a_{ii} - u_i$ for $i = 1, 2$.

The following theorems state formulae for the nucleolus of an arbitrary 2×2 assignment game. In its description we only make explicit the buyers' components, being the sellers' ones easily computed applying (2).

Theorem 3.1. *Let $A \in \mathrm{M}_2^+$ satisfying the normalization conditions (5) and let $d^A = a_{11} + a_{22} - a_{12} - a_{21}$. If*

$$a_{21} > \min\left\{\frac{a_{22}}{2}, \frac{d^A}{2}\right\},$$

the nucleolus of the assignment game is $\nu(w_A) = (u_1^*, u_2^*; v_1^*, v_2^*)$, where $v_i^* = a_{ii} - u_i^*$ for $i = 1, 2$ and *(i)* $(u_1^*, u_2^*) = \left(\frac{a_{11}}{2} + \frac{a_{12}}{2} - \frac{a_{21}}{2},\ \frac{a_{22}}{2}\right),$ *if* $a_{21} \geq a_{12} + a_{22} - a_{11},$

(ii) $(u_1^*, u_2^*) = \left(a_{11} - \left(\frac{a_{21}}{2} + \frac{d^A}{4}\right),\ \frac{a_{21}}{2} + \frac{d^A}{4}\right),$ *if* $a_{21} < a_{12} + a_{22} - a_{11}.$

Proof. Case (i): We have to check that the nucleolus is $\nu(w_A) = (u_1^*, u_2^*; v_1^*, v_2^*) = \left(\frac{a_{11}}{2} + \frac{a_{12}}{2} - \frac{a_{21}}{2},\ \frac{a_{22}}{2};\ \frac{a_{11}}{2} - \frac{a_{12}}{2} + \frac{a_{21}}{2},\ \frac{a_{22}}{2}\right).$

We claim that

$$u_1^* \geq v_1^* \geq u_2^* = v_2^* = \frac{a_{22}}{2} \quad \text{and} \quad v_1^* \geq u_1^* + v_2^* - a_{12} = u_2^* + v_1^* - a_{21} = \frac{d^A}{2}.$$

The first inequality comes from the normalization condition, $a_{12} \geq a_{21}$ and the second one from $a_{21} \geq a_{12} + a_{22} - a_{11}$.

To prove the third inequality, notice that if $\min\left\{\frac{a_{22}}{2}, \frac{d^A}{2}\right\} = \frac{a_{22}}{2}$, we have $a_{21} > \frac{a_{22}}{2}$, and else, if $\min\left\{\frac{a_{22}}{2}, \frac{d^A}{2}\right\} = \frac{d^A}{2}$, we have $a_{22} \geq d^A$ and since $a_{21} \geq a_{12} + a_{22} - a_{11} \geq a_{12} + d^A - a_{11} = a_{22} - a_{21}$, we obtain also that $a_{21} \geq \frac{a_{22}}{2}$. Therefore, in any case, $a_{21} \geq \frac{a_{22}}{2}$, and $v_1^* = \frac{a_{11}}{2} - \frac{a_{12}}{2} + \frac{a_{21}}{2} = \frac{d^A}{2} - \frac{a_{22}}{2} + a_{21} \geq \frac{d^A}{2}$. From our claim, it is immediate to check (6) to (8).

Case (ii): We have to check that the nucleolus is $\nu(w_A) = (u_1^*, u_2^*; v_1^*, v_2^*) = \left(a_{11} - \left(\frac{a_{21}}{2} + \frac{d^A}{4}\right),\ \frac{a_{21}}{2} + \frac{d^A}{4};\ \frac{a_{21}}{2} + \frac{d^A}{4},\ a_{22} - \left(\frac{a_{21}}{2} + \frac{d^A}{4}\right)\right).$

We claim that

$$u_1^* \geq v_2^* \geq u_2^* = v_1^* \geq u_1^* + v_2^* - a_{12} = u_2^* + v_1^* - a_{21} = \frac{d^A}{2}.$$

The first inequality comes from the normalization conditions $a_{11} \geq a_{22}$. The second one comes from $a_{21} < a_{12} + a_{22} - a_{11} = a_{22} - d^A + a_{22} - a_{21}$, and then $a_{21} + \frac{d^A}{2} < a_{22}$.

For the third inequality, notice that in this case $\min\left\{\frac{a_{22}}{2}, \frac{d^A}{2}\right\} = \frac{d^A}{2}$, because if this were not the case, we would have $\frac{d^A}{2} > \frac{a_{22}}{2}$ and $a_{21} > \min\left\{\frac{a_{22}}{2}, \frac{d^A}{2}\right\} = \frac{a_{22}}{2}$, obtaining a contradiction with $a_{21} + \frac{d^A}{2} < a_{22}$. Therefore $a_{21} > \frac{d^A}{2}$. Checking the equalities (6) to (8) is immediate. □

Theorem 3.2. *Let* $A \in \mathrm{M}_2^+$ *satisfying the normalization conditions (5) and let* $d^A = a_{11} + a_{22} - a_{12} - a_{21}$. *If*

$$a_{21} \leq \min\left\{\frac{a_{22}}{2}, \frac{d^A}{2}\right\},$$

the nucleolus of the assignment game is $\nu(w_A) = (u_1^*, u_2^*; v_1^*, v_2^*)$ *, where* $v_i^* = a_{ii} - u_i^*$ *for* $i = 1, 2$ *and*

(i) $(u_1^*, u_2^*) = \left(\frac{a_{11}}{2},\ \frac{a_{22}}{2}\right)$, *if* $a_{12} \leq \frac{a_{22}}{2}$,

(ii) $(u_1^*, u_2^*) = \left(\frac{a_{11}}{2} + \frac{a_{12}}{2} - \frac{a_{22}}{4},\ \frac{a_{22}}{2}\right)$, *if* $\frac{a_{22}}{2} < a_{12} \leq a_{11} - \frac{a_{22}}{2}$,

(iii) $(u_1^*, u_2^*) = \left(a_{11} - \left(\frac{a_{21}}{3} + \frac{d^A}{3}\right),\ \frac{a_{21}}{3} + \frac{d^A}{3}\right)$, *if* $a_{11} - \frac{a_{22}}{2} < a_{12}$.

Proof. Case (i): We have to check that the nucleolus is $\nu(w_A) = (u_1^*, u_2^*; v_1^*, v_2^*) = \left(\frac{a_{11}}{2},\ \frac{a_{22}}{2};\ \frac{a_{11}}{2},\ \frac{a_{22}}{2}\right)$.

Equalities (6) to (8) are immediate to prove, taking into account the normalization conditions (5) and $a_{12} \leq \frac{a_{22}}{2}$.

Case (ii): We have to check that the nucleolus is $\nu(w_A) = (u_1^*, u_2^*; v_1^*, v_2^*) = \left(\frac{a_{11}}{2} + \frac{a_{12}}{2} - \frac{a_{22}}{4},\ \frac{a_{22}}{2};\ \frac{a_{11}}{2} - \frac{a_{12}}{2} + \frac{a_{22}}{4},\ \frac{a_{22}}{2}\right)$.

We claim that

$$u_1^* \geq v_1^* \geq u_2^* = v_2^* = \frac{a_{22}}{2} \quad \text{and} \quad u_2^* + v_1^* - a_{21} \geq v_1^* = u_1^* + v_2^* - a_{12}.$$

The first two inequalities come from $\frac{a_{22}}{2} < a_{12} \leq a_{11} - \frac{a_{22}}{2}$. The third inequality uses that $a_{21} \leq \frac{a_{22}}{2} = u_2^*$. From our claim, it is immediate to check (6) to (8).

Case (iii): We have to check that the nucleolus is $\nu(w_A) = (u_1^*, u_2^*; v_1^*, v_2^*) = \left(a_{11} - \left(\frac{a_{21}}{3} + \frac{d^A}{3}\right),\ \frac{a_{21}}{3} + \frac{d^A}{3};\ \frac{a_{21}}{3} + \frac{d^A}{3},\ a_{22} - \left(\frac{a_{21}}{3} + \frac{d^A}{3}\right)\right)$.

We claim that

$$u_1^* \geq v_2^* \geq u_2^* = v_1^* = u_1^* + v_2^* - a_{12} \quad \text{and} \quad u_2^* + v_1^* - a_{21} \geq v_1^*.$$

The first inequality comes from the normalization condition $a_{11} \geq a_{22}$. The second inequality is a consequence of $a_{11} - \frac{a_{22}}{2} < a_{12}$, which implies $\frac{a_{21}}{3} + \frac{d^A}{3} = \frac{a_{11}+a_{22}-a_{12}}{3} < \frac{a_{22}}{2}$.

For the third inequality, notice that from the hypothesis of the theorem $a_{21} \leq \frac{d^A}{2}$, and therefore we obtain $u_2^* + v_1^* - a_{21} = d^A - \left(\frac{a_{21}}{3} + \frac{d^A}{3}\right) \geq \frac{a_{21}}{3} + \frac{d^A}{3} = v_1^*$. Now, from our claim, equalities (6) to (8) are immediate. □

We can use appropiately the 2×2 case to enlarge the class of assignment games where we can give a formula for the calculation of the nucleolus. For $2 \times m$ (or $m \times 2$) assignment games (for $m > 2$), the computation of the nucleolus can be carried out by reducing appropriately to a 2×2 case. The method is as follows.

Let

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1m} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2m} \end{pmatrix}$$

denote the original matrix. Without loss of generality we may assume that $\mu = \{(1,1), (2,2)\}$ is an optimal matching. Notice that non-assigned sellers get zero at any core allocation.

Let us denote by $p_1^* = \max\{a_{13,}, \ldots, a_{1m}\}$, and $p_2^* = \max\{a_{23,}, \ldots, a_{2m}\}$ and now define a 2×2 assignment matrix in the following way:

$$\widetilde{A} = \begin{pmatrix} \widetilde{a}_{11} & \widetilde{a}_{12} \\ \widetilde{a}_{21} & \widetilde{a}_{22} \end{pmatrix}$$

where $\widetilde{a}_{ij} = \max\{0, a_{ij} - p_i^*\}$ for $i = 1, 2$ and $j = 1, 2$.

We know that $\nu_i(w_A) = \nu_i(w_{\widetilde{A}}) + p_i^*$, for $i = 1, 2$ and $\nu_j(w_A) = \nu_j(w_{\widetilde{A}})$, for $j = 1, 2$ and $\nu_j(w_{\widetilde{A}}) = 0$ for $j = 3, \ldots, m$. This method can be used since non optimally-matched sellers receive zero payoff in any core allocation, and so in the nucleolus. Matrix $\widetilde{A}$ basically represents the reduced assignment game when sellers 3 to m leave the market with zero payoff. For more details see Llerena et al. (2012).

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