Economics Bulletin

Volume 33, Issue 2

The increasing committee size paradox with small number of candidates

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Abstract

The Increasing Committee Size Paradox is a voting inconsistency that occurs under voting systems where voters cast exactly k votes when there are k seats to fill. This paradox occurs when, given an elected committee of size k, one of its member could not be elected if we were to elect a committee of size k + 1; even worse, the two committees may be disjoint. For three-candidate elections, we compute the likelihood of this paradox under the Impartial Anonymous Culture.

Citation: Eric Kamwa, (2013) "The increasing committee size paradox with small number of candidates", *Economics Bulletin*, Vol. 33 No. 2 pp. 967-972.
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Submitted: November 30, 2012. **Published:** April 05, 2013.

1 Introduction

Among the widespread voting systems, there is one under which voters cast exactly $k \geq 1$ votes[1] in order to elect k winners (a committee of k members). This voting system is known as the Limited Voting[2]. With m candidates ($m > k$) in the contest, if $k = 1$, the Limited Voting is equivalent to the Simple Plurality rule; if $k = m - 1$, it is equivalent to the Antiplurality rule. Under the Limited Voting system, if voters are asked to rank candidates strictly *i.e* without ties, Staring (1986) showed that an elected member of a committee of size k could not be elected for a committee of size $k+1$; even worse, the two committees may be disjoint. He called this, the *Increasing Committee Size Paradox*. Using Monte Carlo simulations, Mitchell and Trumbull (1992) evaluated this paradox and they provided the likelihood of the *Increasing Committee Size Paradox* under various probabilistic assumptions with a number of candidates between five and nine with $1 \leq k \leq 4$. The cases with three and four candidates were left out. Does it means that the *Increasing Committee Size Paradox* never occurs with three or four candidates? Maybe no.

In this paper, we compute the likelihood of the *Increasing Committee Size Paradox* in three-candidate elections under the Impartial Anonymous Culture (defined later). This is equivalent to find the probability that an elected candidate for a committee of size one is not elected when the size of the committee grows to two. Due to difficulties encountered in probability computations, we do not evaluate the paradox for four-candidate elections. Nonetheless, we conclude that, with three candidates, the *Increasing Committee Size Paradox* tends to vanish as the electorate grows or tends to infinity.

The rest of the paper is organized as follows : Section 2 is devoted to the definitions of all the needed concepts. In Section 3, we compute the likelihood of the Increasing Committee Size Paradox when the electorate tends to infinity. Section 4 concludes.

2 Preferences and the voting rule

Let N be the set of n voters ($n \geq 2$) and A the set of m candidates, $m \geq 3$. The binary relation R over A is a subset of the cartesian product $A \times A$. For $a, b \in A$, if $(a, b) \in R$, we note aRb to say "a is at least good as b". $\neg aRb$ is the negation of aRb. If we have aRb and $\neg bRa$, we will say "a is better or strictly preferred to b". In this case, we write aPb with P the asymmetric component of R. The symmetric component of R, I, is defined by aIb translating an indifference between a and b *i.e* $\neg aRb$ and $\neg bRa$. The preference profile $\pi = (P_1, P_2, ..., P_i, ..., P_n)$ gives all the linear order[3] of all the n voters on A where P_i is the strict ranking of a given voter i. The set of all the preference profiles of size n on A is denoted by $P(A)^n$. In the sequel, we will write *abc* to say that a is strictly preferred to b and b strictly preferred to c. A voting situation $\tilde{n} = (n_1, n_2, ..., n_t, ..., n_{m!})$ indicates the number

[1] Generally, for their k top or preferred candidates.

[2] For more on the variants of this voting system, see Dummett (1984).

[3] A linear order is a binary relation that is transitive, complete and antisymmetric. The binary relation R on A is *transitive* if for $a, b, c \in A$, if aRb and bRc then aRc. R is *antisymmetric* if for all for $a \neq b$, $aRb \Rightarrow \neg bRa$; if we have aRb and bRa, then $a = b$. R is *complete* if and only if for all $a, b \in A$, we have aRb or bRa.

Table 1: Voting situation $\tilde{n}$ for $A = \{a, b, c\}$

$n_1 : abc$	$n_2 : acb$	$n_3 : cab$
$n_4 : cba$	$n_5 : bca$	$n_6 : bac$

of voters for each linear order such that $\sum_{t=1}^{m!} n_t = n$. Table 1 describes the voting situation with three candidates.

Suppose that we want to elect a committee with k members ($1 \leq k < m$). Let $r(i, a)$ be the rank of candidate a in voter i's ranking. If voters are assumed to vote for their k top candidates, the scoring vector associated to the voting system is $w = (w_1, w_2, w_3, ..., w_k, ..., w_m)$ such that $w_1 = ... = w_k = 1$ and $w_{k+1} = ... = w_m = 0$. If $k = 1$, we have the Simple Plurality rule; if $k = m - 1$, we have the Antiplurality rule. The candidate a's score is given by

$$W(\pi, a) = \sum_{i=1}^{n} w_{r(i,a)}$$

Hereafter, we will simply write $W_k(a)$ the candidate a's score given k. The elected candidates are those with the k greatest scores. We denote by C^k the set of all possible committees of size k and by $C_k^* \in C^k$ the elected committee given k. For a given voting situation, it may exist more that one elected committee when two or more candidates tied at the *k-th* greatest score. The tie can be broken by a lottery to preserve anonymity and neutrality[4] or it could be assumed that the electorate is too large in a manner that the probability of a tie vanishes. In this paper, the tie will always be broken in order to favor the paradox.

With three candidates, $w = (1, 0, 0)$ for $k = 1$ and $w = (1, 1, 0)$ for $k = 2$. On $A = \{a, b, c\}$, with $C_1^* = \{a\}$, the Increasing Committee Size Paradox occurs only if $C_2^* = \{b, c\}$. The original example that illustrates the Increasing Committee Size Paradox was given by Staring (1986). Let us recall that example.

Example 1. *Consider a voting situation with 12 voters wishing to elect a committee of two members among 9 candidates, a, b, c, d, e, f, g, h and z. Preferences were only the four first preferred candidates are listed as follows :*

$2 : afch...$	$1 : agch...$	$1 : bgdz...$	$2 : bhdf...$
$1 : chef...$	$1 : czeg...$	$1 : dzed...$	$1 : dafz...$
$1 : ebgz...$	$1 : ezga...$		

Given k, the candidates scores are as follows:

	a	b	c	d	e	f	g	h	z
$k = 2$	4	4	2	2	2	2	2	3	3
$k = 3$	4	4	5	5	5	3	4	3	3
$k = 4$	5	4	5	5	5	6	6	6	6

[4] The property of *anonymity* states that in a voting situation, names of voters are irrelevant; while the name of candidates are irrelevant under the property of *neutrality*.

Thus, $C_2^* = \{a, b\}$, $C_3^* = \{c, d, e\}$ *and* $C_4^* = \{f, g, h, z\}$. *It follows that* $C_2^* \cap C_3^* = \emptyset$ *and* $C_3^* \cap C_4^* = \emptyset$. *Each supposed increase in the committee size lead to the strong version of the Increasing Committee size paradox.*

3 The likelihood of the Increasing Committee Size Paradox

We first state the conditions for the Increasing Committee Size Paradox to occur in three-candidate elections and for committees of size two.

Proposition 1. *For* $A = \{a, b, c\}$ *and* $k = 1$ *such that* $C_1^* = \{a\}$, *the Increasing Committee Size Paradox occurs when the committee size grows to* $k = 2$ *if and only if*

$$\begin{cases} W_1(a) \geq W_1(b) \\ W_1(a) \geq W_1(c) \\ W_2(b) \geq W_2(a) \\ W_2(c) \geq W_2(a) \end{cases} \Leftrightarrow \begin{cases} n_1 + n_2 \geq n_5 + n_6 \\ n_1 + n_2 \geq n_3 + n_4 \\ n_4 + n_5 \geq n_2 + n_3 \\ n_4 + n_5 \geq n_1 + n_6 \end{cases}$$

The proof of proposition 1 is left to the reader since the conditions come simply from the definition of the Increasing Committee Size Paradox.

Fitted with this proposition, we are able to compute the likelihood of the considered paradox. For this, we use the Impartial Anonymous Culture assumption (IAC). Under IAC, each voting situation is equally likely. The likelihood of a given voting situation is described by a multinomial distribution:

$$Prob(\tilde{n} = (n_1, ..., n_t, ..., n_{m!})) = \frac{1}{C^n_{n+m!-1}} = \frac{n!(m!-1)!}{(n+m!-1)!}$$

with $C^n_{n+m!-1}$ the total number of possible voting situations. For more details about IAC, see among others Gehrlein and Fishburn (1976), Huang and Chua (2000), etc. The likelihood of the Increasing Committee Size Paradox (ICSP) will be calculated in respect with the following ratio:

$$\frac{\text{Number of voting situations in which the ICSP is likely}}{\text{Total number of possible voting situations}}$$

To make all our calculations, we use the computer program designed by Moyouwou[5] in Maple code. This computer program is based on Mbih *et al.* (2006)'s works and follows the same techniques as Gehrlein and Fishburn (1976), Huang and Chua (2000).

We denote by $P_{m,n}(k, k+1)$, the probability that the Increasing Committee Size Paradox occurs with m candidates, n voters when the size of the committee grows from k to $k+1$. Proposition 2 gives the two first polynomials obtained with the computer program for $m = 3$ and $k = 1$.

[5]For this, we are very grateful to I. Moyouwou. The computer program is available on demand.

Proposition 2. *With three candidates and at least seven voters, the likelihood of the Increasing Committee Size Paradox is:*

$$P_{3,n\equiv 0[72]}(1,2) = \frac{5}{108}\,\frac{(n+6)\,(2\,n^4+35\,n^3+222\,n^2+756\,n+1296)}{(n+5)\,(n+4)\,(n+3)\,(n+2)\,(n+1)}$$

$$P_{3,n\equiv 1[72]}(1,2) = \frac{5}{108}\,\frac{(n+11)\,(n-1)\,(2\,n+7)\,(n+5)}{(n+4)\,(n+3)\,(n+2)\,(n+1)}$$

where $n \equiv x[y]$ means n and x are congruent modulo y.

Doing computations with all the polynomials obtained, table 2 summarizes the frequencies for various number of voters.

4 Conclusion

According to table 2, with three candidates an increase of the size of a committee from 1 to 2, leads to the *Increasing Committee Size Paradox* (ICSP) in 15.90% of the cases with only seven voters. As the number of voters tends to infinity, likelihood of the ISCP falls to 0.0925. Thus, one cannot say that because the three-candidate case was left out by Mitchell and Trumbull (1992) means that the ISCP could not occur. Our computations show that the likelihood of the ISCP is not negligible with three candidate. What about the four-candidate case? We try computations with four candidates but we were not able to provide accurate results with the computer program due to the difficulties encountered in the computations. Nonetheless, we found that, the limit probability of the ISCP with four candidates when the size of the committee goes from 2 to 3, is equal to zero. So, with four candidates, the likelihood of the ISCP tends to vanish as the electorate grows.

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Table 2: Likelihood of the ISCP with three candidates

n	$P_{3,n}(1,2)$
1	
2	
3	
4	
5	
6	
7	0.1590
8	0.1981
9	0.1798
10	0.1528
11	0.1634
12	0.1619
13	0.1386
14	0.1509
15	0.1455
16	0.1335
17	0.1389
18	0.1378
19	0.1270
20	0.1329
21	0.1304
22	0.1237
23	0.1269
24	0.1261
25	0.1248
:	:
:	:
72	0.1035
:	:
144	0.0980
:	:
288	0.0953
:	:
:	:
∞	0.0925