Economics Bulletin

Volume 33, Issue 3

Why do dictators like white elephants? An application of the all-pay auction.

Marco Magnani
Department of Economics, Università degli Studi di Parma

Abstract

The present paper studies the provision of big and inefficient public investments in proprietary states. By means of these investments, whose rate of return is either nil or negative, an incumbent dictator can transfer resources to his/her supporters and obtain a head-start advantage in the contest with a challenger. These projects are realized only if they are not too inefficient and more importantly, big enough to provide a large enough head-start advantage.

Citation: Marco Magnani, (2013) "Why do dictators like white elephants? An application of the all-pay auction.", *Economics Bulletin*, Vol. 33 No. 3 pp. 2260-2269.
Contact: Marco Magnani - marco.magnani@unipr.it.
Submitted: August 30, 2013. **Published:** September 05, 2013.

1 Introduction

Resource misallocation has been recognized as one of the main problems affecting economic growth in developing countries. This is documented by Hall and Jones, (1999) who find that there is a large gap in total factor productivity between rich and poor countries in a data-set including 127 countries.

Public investments are often held to be responsible for resource misallocation. In a recent paper, Keefer and Knack (2007) show that public investment is dramatically higher in countries with no competitive elections or with low-quality governance and limited political checks and balances. They argue that in these countries public investments have the aim of increasing politician rent and fund largely unproductive projects which are mainly intended to direct rents to government officials or their cronies.

Many authors also consider corruption and weak institutions as a key determinant of the resource curse, i.e. the paradoxical evidence that resource-rich countries tend to grow more slowly than resource-poor ones[1].

A clear example of resource misallocation and misuse of public investment are so-called "white elephants", public projects having three main characteristics: that they are big, expensive and inefficient. The most documented cases of "white elephants" in the literature are that of INDECO, the Industrial Development Corporation of Zambia whose poor effects on the country development are studied by Tangri (1999), and those included in Killick's book (1978) which focuses on the Ghanaian economy and is one of the most detailed studies on this issue[2].

Political economy theory has tried to explain why politicians invest in inefficient projects by focusing mostly on the incentives of candidates competing in elections. In this context, Coate and Morris (1995) and Lizzeri and Persico (2001) show that pork-barrel spending (which includes "white elephants"), is a form of inefficient redistribution which allows politicians to raise the income of particular constituencies. In other words, it is a means of targeting funds at specific sections of the electorate with the aim of increasing the chances of a candidate of winning elections. Robinson and Torvik (2005) moreover show that social inefficiency is an essential feature for "white elephants" to become a credible instrument through which an incumbent politician can redistribute resources toward specific groups of voters.

The present paper contributes to this literature by considering the case of large inefficient public investments from a different perspective. It is hard to describe certain developing countries as democracies; rather they are proprietary states which are ruled by dictators. "White elephants" are also used in dictatorships and is perhaps misleading to consider elections as a rationale. The focus here is thus on proprietary states and on the reasons why a dictator invests in inefficient projects.

What is particular to proprietary states is the absence of defined property rights over the rents generated by running the country, which accrue to the incumbent dictator. This severely affects the amount and the type of investment made.

The seminal paper by Konrad (2002) studies the provision of efficient investments in the absence of well-defined property rights. He shows that in facts investing in projects which increase the revenues accruing to an incumbent agent poses two problems. First it increases the number of potential challengers. Second, when a contest occurs, it creates the incentive for rivals to make more effort. In this context,

[1] See Van der Ploeg (2011) for a complete survey of this issue.

[2] Other papers deal indirectly with "white elephants" when discussing the poor performances of public investments in promoting economic growth. On this point see for instance Gylfason (2001) and Auty (2001) who study the case of oil exporting countries.

incumbency plays a key role. If the agent has an incumbency advantage under-investment occurs. In absence of an incumbency advantage, no welfare improving investments are realized at all.

This reasoning can be extended further to a case which could be defined "negative" investment. Allen (2002) shows that in the absence of property rights the owner of a valuable item might lower its value intentionally in order to lower the probability of conflict with an encroacher. He reports several examples ranging from rhino dehorning to the construction of penal colonies in colonial territories which reduce their attractiveness. Gonzalez (2005) also shows that technological backwardness may be intentionally pursued with the aim of reducing the size of the contested resources.

A further circumstance is analyzed here, which differs from those cited above in the specific nature of the investments considered. In particular, in the present framework, the public project has no effects, either positive or negative, on the future rents accruing to an incumbent dictator. The investment is only meant to transfer part of the present rents to the cronies[3]. As such the "white elephant" is intrinsically inefficient because redistribution happens through a "leaky bucket", and its costs exceed the actual size of the transfer.

The investment is merely "defensive" and provides the incumbent dictator with an advantage in the contest with a challenger. This advantage can take many forms ranging from electoral and financial to military support. No restrictions are introduced into the model, which simply assumes that the size of the advantage is somehow proportional to the amount of funds accruing to the cronies of the dictator. This seems reasonable given that any of the above-mentioned forms of support has a financial cost[4].

In this context, as in Konrad (2002), incumbency is a main issue and its importance is captured by the following sentence by Robinson and Torvik who write: "White elephants are (...) part of an exchange relationship between politicians and voters (a situation which political scientists call ‘clientelism’) where there are important advantages of incumbency" (Robinson and Torvik, 2005, pp. 201). This argument refers to the case of elections but it is easily extended to proprietary state where clientelism is a widespread phenomenon.

In the present framework, incumbency takes the form of an investment opportunity available only to the dictator in power. The incumbent who knows that he/she will participate in the future in a contest with a rival, considers spending resources by which he can obtain a head-start advantage and enhance his/her probability of winning.

Two main results are obtained. First it is shown that an incumbent dictator invests in inefficient projects if their rate of return is not too negative. Second it is proven that only big projects are chosen. This confirms that such inefficient investments are in fact "white elephants".

2 The Model

The following two-period framework is considered. At the beginning of Period 1 the dictator in power (the incumbent, "he"), is in charge of making an investment decision. At the end of Period 1, the investment pays off and an encroacher (the

[3]Note that there are no opportunity costs either. The resources involved in the redistribution process would be appropriated by the dictator if they were not transferred to his/her supporters

[4]It is worth stressing that the advantage for the dictator does not come from the investment itself, as for instance, in the case of a military facility, nor from the monetary pay-off which it provides. On the contrary, the public project has the purpose to win the loyalty of a share of the population which can be used in the contest with a challenger.

rival, "she") shows up who challenges the incumbent in a contest. The agent who prevails in the contest is the incumbent in period 2 and obtains rent V.

The investment decision involves three options. The incumbent can choose between a big and a small project and can also decide not to invest. Both types of project have the same probability of success, which for the sake of simplicity is $\frac{1}{2}$[5]. A successful investment provides the incumbent with a head-start advantage in the contest. If the investment is not successful, no advantage is generated. The choice over the investment is observed by the rival, but its outcome is private information of the incumbent. The probability that an investment is successful is common knowledge[6].

The contest for incumbency is described as a first price all-pay auction. The incumbent and the rival simultaneously exert contest efforts x^I and x^R, respectively. The contest success function:

$$q\left(x^I, x^R\right) = \begin{Bmatrix} 1 \\ \frac{1}{2} \\ 0 \end{Bmatrix} if \begin{Bmatrix} x^I + k > x^R \\ x^I + k = x^R \\ x^I + k < x^R \end{Bmatrix} \tag{1}$$

defines the incumbent's probability of winning which depends on contest efforts. If $x^I + k > x^R$ the incumbent wins while if $x^I + k < x^R$, he loses. A coin is tossed to determine who wins if a tie occurs. The rival wins with probability $1 - q\left(x^I, x^R\right)$.

If the incumbent does not invest in Period 1, $k = 0$ holds and the contest is symmetrical. If an investment is realized, two asymmetries emerge. The first asymmetry derives from the fact with probability $\frac{1}{2}$, $k > 0$ holds and the incumbent has a head-start advantage in the contest. The second asymmetry is informational: only the incumbent observes the state of the world and knows if a head-start is actually generated or not.

At the end of Period 1 the incumbent chooses his contest effort x^I to maximize the expected discounted sum of net revenues. Assuming for the sake of simplicity that the inter temporal discount factor is nil, his payoff is:

$$p^I = -C - x^I + q\left(x^I, x^R\right) \cdot V \tag{2}$$

where C denotes the investment cost.

If no investment is realized, $C = 0$ holds, while it is $C_b = \frac{\alpha_b(1+\xi)}{2}$ and $C_s = \frac{\alpha_s(1+\xi)}{2}$ respectively in cases where the incumbent invests in a big or a small project. Investment costs reduce the incumbent rent in Period 1[7] and are proportional to the size of the expected head-start advantage generated in each case[8]: $E[k_b] = \frac{\alpha_b}{2}$ and $E[k_s] = \frac{\alpha_s}{2}$. The proportionality factor $(1 + \xi)$ depends on the parameter ξ which measures inefficiency. Since $\xi \geq 0$ holds, the investment rate of return is either nil or negative.

The first type of project, big, gives a large head-start advantage, $\alpha_b \geq V$ and is in fact a "white elephant". The second project is a small investment which provides an advantage $0 < \alpha_s < \frac{V}{2}$[9].

[5] Considering a generic probability λ gives qualitatively similar results.

[6] A successful investment benefits the targeted segment of the population and wins its support. On the contrary, an investment is not successful if for instance, benefits are wasted, diverted or appropriated by a small group within the targeted segment of the population. This piece of information is easily observed only by the dictator in office.

[7] For instance the investment is financed by the fiscal revenues which the incumbent dictator gathers in period. 1.

[8] The size of the advantage is measured in money.

[9] This assumption is required for model tractability because no equilibrium exists for a parameter constellation where $V > \alpha_s \geq \frac{V}{2}$.

Consider now the rival's payoff function which is simply:

$$p^R = -x^R + \left[1 - q\left(x^I, x^R\right)\right] V. \tag{3}$$

The rival exerts contest effort x^R and wins with probability $1 - q\left(x^I, x^R\right)$. In this case, she becomes Period-2 incumbent and obtains rent V.

3 Equilibrium Analysis

The game is sequential and the characterization of the equilibrium makes it necessary to use backward induction. The second stage of the game where the incumbent and the rival participate in the contest (the contest stage from now on) is thus considered first.

Different equilibria emerge depending on the investment of the incumbent in the first stage. If the investment is not realized the contest stage is a standard all-pay auction with complete information and two symmetrical players. The equilibrium in this case, is well known in the literature, and is characterized in the following proposition, derived by Hillman and Riley (1989).

Lemma 1 (Hillman and Riley, 1989) *If $k = 0$, in the unique equilibrium for the all-pay auction, both the incumbent and the rival randomize their contest efforts over the support $[0, V]$ according to the cumulative distribution function:*

$$F\left(x^i\right) = \frac{x^i}{V} \tag{4}$$

with $i = \{I, R\}$, and obtain a nil payoff.

Consider now the circumstance where an investment is realized in Period 1. In this case the contest is a first-price all-pay auction with asymmetrical players and a random head-start. The size of the advantage awarded to the incumbent is in fact uncertain, and he has private information on it.

Denote by $F^I_\alpha\left(x^I_\alpha\right)$ and $F^I_0\left(x^I_0\right)$ the cumulative distribution functions for the effort of the incumbent respectively when he has the head-start advantage and when he has not. Denote by $F^R\left(x^R\right)$ the distribution of the effort of the rival. The expected payoffs for the contest stage are defined as follows:

$$E\left[U^I\left(x^I_\alpha, F^R\left(\alpha + x^A_\alpha\right)\right)\right] = V \cdot F^R\left(\alpha + x^I_\alpha\right) - x^I_\alpha = p^I_\alpha \tag{5}$$

$$E\left[U^I\left(x^I_0, F^R\left(x^I_0\right)\right)\right] = V \cdot F^R\left(x^I_0\right) - x^I_0 = p^I_0 \tag{6}$$

$$E\left[U^R\left(x^R, F^I_\alpha\left(x^R - \alpha\right), F^I_0\left(x^R\right)\right)\right] = \frac{V}{2} \cdot F^I_\alpha\left(x^R - \alpha\right) + \frac{V}{2} \cdot F^I_0\left(x^R\right) - x^R = p^R \tag{7}$$

where $\alpha \in \{\alpha_b, \alpha_s\}$.

In this context, the choice to invest in a big or in a small project defines different equilibria. Consider first the case where the big project is chosen. The following proposition characterizes the equilibrium.

Proposition 2 *If $k = \alpha_b$ in the unique equilibrium for the contest stage the incumbent:*

- *exerts a nil effort with probability 1 when he has the head-start advantage, and obtains $p^I_\alpha = V$;*

- *randomizes his contest effort over the support* $\left[0, \frac{V}{2}\right]$ *according to the cumulative distribution function:*

$$F_0^I\left(x_0^I\right) = \frac{2 \cdot x_0^I}{V} \tag{8}$$

when he does not have the head-start advantage, and obtains $p_0^I = \frac{V}{2}$.

The rival randomizes her contest effort over the support $\left[0, \frac{V}{2}\right]$ *according to the cumulative distribution function:*

$$F^R\left(x^R\right) = \frac{1}{2} + \frac{x^R}{V} \tag{9}$$

and obtains $p^R = 0$.

Proof. Consider the state of the world where the advantage is generated. An effort level $x^R \geq \alpha_b > V$ gives a strictly negative payoff and is dominated by $x^R = 0$. This further implies that the payoff of the incumbent, if he exerts the effort $x_\alpha^I = 0$, is $p_\alpha^I = V$, and every $x_\alpha^I > 0$ is a dominated action.

Given these results, the contest is strategically equivalent to an all-pay auction with complete information and no head-start advantage where the valuations of the prize are respectively, $\frac{V}{2}$ for the rival and V for the incumbent. This is easily seen if p_0^I and p^R are considered. Note in fact that p_0^I is defined as in Equation 6, while the expected payoff of the rival becomes:

$$E\left[U^R\left(x^R, F_0^I\left(x^R\right)\right)\right] = \frac{V}{2} \cdot F_0^I\left(x^R\right) - x^R = p^R. \tag{10}$$

The results by Hillman and Riley (1989) and Baye et al. (1996) apply which allow us to characterize the equilibrium strategies and payoffs and proving uniqueness of the equilibrium. ■

If the head-start advantage is generated, the rival never prevails in the contest. Hence the incumbent participates in the auction only if he has no head-start advantage.

Consider now what happens if a small project is chosen. Before proceeding to the equilibrium analysis some definitions need to be introduced. Following Siegel (2009), define $x^I = x_0^I = x_\alpha^I - \alpha_s$ as the incumbent effort score. Define:

$$\Phi^I\left(x^I\right) = \frac{1}{2}\left[F_0^I\left(x_0^I\right) + F_\alpha^I\left(x_\alpha^I - \alpha_s\right)\right] \tag{11}$$

as the aggregate cumulative distribution function for the expected effort score of the incumbent over the two possible states of nature. The equilibria for the contest stage are summarized in Proposition 3.

Proposition 3 *If* $k = \alpha_s$ *the contest stage has a continuum of equilibria characterized by the following elements.*

The incumbent randomizes his effort scores over the support $[0, V]$ *according to the aggregate cumulative distribution function:*

$$\Phi^I\left(x^I\right) = \frac{x^I}{V} \tag{12}$$

and obtains $p_\alpha^I = \alpha_s$ *and* $p_0^I = 0$ *respectively when he has and when he does not have the head-start advantage.*

The rival randomizes her contest effort over support $[0, V]$ *according to the cumulative distribution function:*

$$F^R\left(x^R\right) = \frac{x^R}{V} \tag{13}$$

and obtains $p^R = 0$.

Proof. See the Appendix. ■

In this case, the incumbent always participates in the contest, but obtains a positive payoff only when he has the head-start.

Consider now the first stage of the game where the incumbent decides the size of the investment, $C \in \{0, C_b, C_s\}$. The following result holds.

Proposition 4 *The incumbent does not invest in the small project and* $C = C_s$ *is never an equilibrium for the game.*

$C = C_b$ *is an equilibrium for the game if* $\xi \leq 1 - \frac{3}{2} \cdot \frac{V}{\alpha_b}$ *and* $\frac{2}{3} \cdot \alpha_b \leq V$ *hold.*

$C = 0$ *is an equilibrium for the game if* $\xi > 1 - \frac{3}{2} \cdot \frac{V}{\alpha_b}$ *or if* $\frac{2}{3} \cdot \alpha_b > V$ *hold.*

Proof. Note initially that if $C = C_s$, by Proposition 3:

$$E\left[p^I\right] = -C_s + \frac{1}{2}\left(p_0^I + p_\alpha^I\right) = -\frac{\alpha_s \cdot \xi}{2} < 0 \tag{14}$$

On the other hand, if $C = 0$, by Proposition 1, $E\left[p^I\right] = 0$ holds, implying that $C = C_s$ is a dominated strategy. Consider now the expected payoff of the incumbent if $C = C_b$:

$$E\left[p^I\right] = -\alpha_b(1 - \xi) + \frac{3}{2} \cdot V. \tag{15}$$

In order for the big investment to be chosen in equilibrium, $E\left[p^I\right] \geq 0$ must hold i.e.:

$$\frac{3}{2} \cdot V \geq \alpha_b(1 - \xi) \tag{16}$$

which further requires:

$$\xi \leq 1 - \frac{3}{2} \cdot \frac{V}{\alpha_b}$$

Note lastly that since by assumption $\xi \geq 0$, if

$$\frac{2}{3} \cdot \alpha_b \leq V \tag{17}$$

holds, no investment is provided. ■

In the case of a "white elephant" the incumbent obtains a positive payoff if the degree of inefficiency of the investment is not too high and if the project is not too big compared to the contested rent.

4 Final Remarks

The analysis studies the provision of "white elephants" in proprietary states. These investments are vehicles through which an incumbent dictator may transfer resources to supporters, and as such, they have a rate of return which is either nil or negative. Project inefficiency is thus not, as in Robinson and Torvik (2005), an essential feature for "white elephants" to be a credible redistribution tool. Rather it is an intrinsic characteristic of the investment. In this context, an incumbent dictator is willing to realize these projects only if they are not too inefficient and more importantly, big enough to provide a large enough head-start advantage in the contest with a challenger.

References

[1] Allen DW (2002) The Rhino's Horn: Incomplete Property Rights and the Optimal Value. *Journal of Legal Studies* 31(S2): 339-358.

[2] Auty RM (2001) *Resource Abundance and Economic Development*. Oxford: Oxford University Press.

[3] Baye R, Kovenock D, de Vries CG (1996) The All-Pay Auction with Complete Information. *Economic Theory*, 8(2): 291-305.

[4] Che YK, Gale IL (1998) Caps on Political Lobbying. *American Economic Review* 88(3): 643-651.

[5] Coate S, Morris S (1995) On the Form of Transfers to Special Interests. *Journal of Political Economy* 103: 1210-1235.

[6] Ellingsen T (1991) Strategic Buyers and the Social Cost of Monopoly. *American Economic Review* 81(3): 648-657.

[7] Gonzalez FM (2005) Insecure Property and Technological Backwardness. *The Economic Journal* 115(505): 703-721.

[8] Gylfason T (2001) Natural Resources, Education, and Economic Development. *European Economic Review* 45(4): 847-859.

[9] Hall RE, Jones IJ (1999) Why Do Some Countries Produce So Much More Output Per Worker than Others?. NBER Working Paper nr. 6564.

[10] Hillman A, Riley J (1989) Politically Contestable Rents and Transfers. *Economics and Politics* 1(1): 17-39.

[11] Keefer P, Knack S (2007) Boondoggles, Rent-Seeking, and Political Checks and Balances: Public Investment under Unaccountable Governments. *The Review of Economics and Statistics* 89(3): 566-572.

[12] Killick T (1978). *Development Economics in Action: A Study of Economic Policies in Ghana*. London: Heinemann.

[13] Konrad K (2002) Investment in the Absence of Property Rights: the Role of Incumbency Advantages. *European Economic Review* 46(8): 1521-1537.

[14] Lizzeri A, Persico N (2001) The Provision of Public Goods under Alternative Electoral Incentives. *American Economic Review* 91: 225-245.

[15] Robinson JA, Torvik R (2005) *White Elephants*. Journal of Public Economics 89(2): 197- 210.

[16] Siegel R (2009) All-pay contests. *Econometrica* 77(1): 71-92.

[17] Tangri R (1999) *The Politics of Patronage in Africa*. Oxford: James Currey.

[18] Van der Ploeg F (2011) Natural Resources: Curse or Blessing?. *Journal of Economic Literature* 49(2): 366-420.

5 Appendix: Proof of Proposition 3

Some preliminary results are worth mentioning.

Lemma 5 *The functions* $\Phi^I\left(x^I\right)$ *and* $F^R\left(x^R\right)$ *are continuous and atoms of probability are only placed in* 0.

Proof. Following Hillman and Riley (1989) [10] it is possible to exclude that there is a value for the expected effort score of the incumbent $\kappa \in (0, V]$ which is exerted with strictly positive probability. But if there is a value exerted with strictly positive probability, the rival's probability of winning rises discontinuously at $x^R = \kappa$ and there is some $\varepsilon > 0$ such that the rival exerts the contest effort $[\kappa - \varepsilon, \kappa]$ with nil probability. The incumbent is thus better off shifting the mass of probability down

[10] See also Baye et al. (1996), Che and Gale (1998) and Ellingsen (1991).

from κ to $\kappa - \varepsilon$ to reduce his expected effort score without affecting the probability of winning. Hence $\Phi^I\left(x^I\right)$ must be continuous. A symmetrical argument also proves the continuity of $F^R\left(x^R\right)$. Moreover since $x^I \geq 0$ and $x^R \geq 0$ hold, atoms of probability in $\Phi^I\left(x^I\right)$ and $F^R\left(x^R\right)$ can only be placed in 0. ■

Define now $\underline{x}_j^I$ and $\underline{x}^R$ as the lower bounds respectively for the contest effort distribution of the incumbent in the state of the world j (with $j = \{0, \alpha\}$) and for the contest effort distribution of the rival. In the same way define the upper bounds $\bar{x}_j^I$ and $\bar{x}^R$. Further results are the following.

Lemma 6 *In equilibrium* $\underline{x}_0^I = \underline{x}^R = 0$ *must hold.*

Proof. Suppose initially that $\underline{x}_0^I < \underline{x}^R$. Any effort level, x_0^I, such that $\underline{x}_0^I \leq x_0^I < \underline{x}^R$ gives the incumbent a nil probability of winning and a negative payoff. It is thus strictly dominated by $x_0^I = 0$ implying that $\underline{x}_0^I < \underline{x}^R$ never holds in equilibrium and $\underline{x}_0^I \geq \underline{x}^R$. Analogously it is possible to show that $\underline{x}_\alpha^I + \alpha_s \geq \underline{x}^R$ holds. A symmetrical argument further requires that the inequality

$$\min\left\{\underline{x}_0^I, \underline{x}_\alpha^I + \alpha_s\right\} \leq \underline{x}^R \tag{18}$$

is verified in equilibrium.

Note now that $\underline{x}^R = 0$ must hold. In fact, if $\underline{x}^R > 0$, the rival has the profitable deviation to shift down the lower bound of the distribution. This does not affect the probability of winning, since by Lemma 5 atoms of probability are only placed in 0 and the inequalities $\underline{x}_0^I \geq \underline{x}^R$ and $\underline{x}_\alpha^I + \alpha_s \geq \underline{x}^R$ hold, but reduces the expected contest effort of the rival. As a consequence, by Inequality 18, and since $\underline{x}_\alpha^I + \alpha_s > 0$ and $\underline{x}_0^I \geq \underline{x}^R$ hold, in equilibrium it must also be the case that $\underline{x}_0^I = \underline{x}^R$. ■

Lemma 7 *No equilibrium strategies exist such that* $p_0^I > 0$ *and* $p^R > 0$ *hold.*

Proof. Lemma 6 establishes that in equilibrium $\underline{x}_0^I = \underline{x}^R = 0$ holds. Note that $p_0^I > 0$ requires $F^R(0) > 0$ since it is $V \cdot F^R(0) > 0$ when $x_0^I = 0$. The same argument establishes that $p^R > 0$ requires $F_0^I(0) > 0$. If $p_0^I > 0$ and $p^R > 0$ hold at the same time, a profitable deviation exists since the probability of obtaining the prize increases by a finite amount, if either the incumbent or the rival choose an effort level slightly larger than 0. ■

Note that in equilibrium $p^R = 0$ holds. But suppose this is not the case. If $p^R > 0$, by Lemma 7, $p_1^0 = 0$ holds and $F^R\left(x^R\right)$ must have an upper bound $\bar{x}^R < V$. The incumbent thus has the profitable deviation to submit a bid $x_0^I \in \left(\bar{x}^R, V\right)$ and get a positive payoff when he does not have the head-start.

The condition $p^R = 0$ requires the following equality to be verified for every $x^R \in [0, \alpha_s)$:

$$\frac{V}{2} \cdot F_0^I\left(x^R\right) - x^R = 0$$

so that[11]:

$$F_0^I\left(x_0^I\right) = \frac{2 \cdot x_0^I}{V}$$

and

$$\Phi^I\left(x^I\right) = \frac{x^I}{V}$$

[11] Note that $F_0^I(\alpha_s) < 1$ by the assumption $\alpha_s < \frac{V}{2}$.

Consider now the equilibrium conditions for the interval $[\alpha_s, V]$. It must be the case that:

$$\begin{aligned} & \frac{V}{2} \cdot F_0^I(\alpha_s) + \frac{V}{2}\left[1 - \frac{1}{2}F_0^I(\alpha_s)\right] \Pr\left[x^R - \alpha_s > x_\alpha^I | x^R \geq \alpha_s\right] + \\ & + \frac{V}{2}\left[1 - \frac{1}{2}F_0^I(\alpha_s)\right] \Pr\left[x^R > x_0^I | x^R \geq \alpha_s\right] - x^R \\ = & \, 0 \end{aligned}$$

or

$$V \cdot \Phi^I\left(x^R\right) - x^R = 0$$

so that

$$\Phi^I\left(x^I\right) = \frac{x^I}{V}.$$

By the continuity of the equilibrium strategies established in Lemma 5, $\Phi^I\left(x^I\right)$ is a uniform over the support $[0, V]$. This further implies that there is a continuum of equilibria. If $x^I > \alpha_s$ any pair $F_0^I\left(x_0^I\right)$ and $F_\alpha^I\left(x_\alpha^I - \alpha_s\right)$ such that $\Phi^I\left(x^I\right) = \frac{x^I}{V}$, is in fact an equilibrium of the contest stage.

Note that in equilibrium $\Phi^I(V) = 1$ must hold. If this is not the case, the rival has the profitable deviation to exert an effort $V > x^R > \bar{x}^I$ and obtain a positive payoff. This further implies that $\bar{x}_\alpha^I = V - \alpha_s$ and $\bar{x}_0^I = V$. Hence the following conditions hold respectively for every $x_0^I \in [0, V]$ and every $x_\alpha^I \in [0, V - \alpha_s]$:

$$V \cdot F^R\left(x_0^I\right) - x_0^I = 0$$

and

$$V \cdot F^R\left(x_\alpha^I + \alpha_s\right) - x_\alpha^I = \alpha_s$$

so that:

$$F^R\left(x^R\right) = \frac{x^R}{V}$$

By the result of Lemma 5, the rival randomizes uniformly over the support $[0; V]$.