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Effect of pollution on the total factor productivity and the Hopf bifurcation

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Abstract

In a recent contribution, Empora and Mamuneas (2011) find a positive relation between pollution emissions and the total factor productivity (TFP). In the present paper, we show that this positive effect reduces the effect of pollution on the marginal utility of consumption for which a limit cycle occurs.

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1 Introduction

Pollution can affect both the household's preferences and the technology. The effect of pollution on the household's utility is widely explored in the literature. Since Heal (1982) and Michel and Rotillon (1996), it is well-known that a positive pollution effect on the marginal utility of consumption may lead to a limit cycle through a Hopf bifurcation. Using an endogenous growth framework, very close to Benhabib and Farmer (1994), Itaya (2008) shows also that the non-separability between consumption and pollution in the utility function enhances the range of parameters for which local indeterminacy occurs.

However, the effect of pollution on the production sector seems to be neglected. From a theoretical point of view, the relation between pollution and growth is ambiguous. On one hand, pollution can decrease growth through its negative effect on health and in turn on worker productivity (Schlenker and Walker (2011) and Graff Zivin and Neidell (2012)). On the other hand, pollution can increase growth through its relation with inputs of production (Diao and Roe (1997)). To the best of our knowledge, one of the first empirical investigation of the effect of pollution on the total factor productivity is Empora and Mamuneas (2011). Using a data set for 48 contiguous U.S States for the period 1965-2002, they found that air pollution, namely sulphur dioxide (SO2) and nitrogen oxide (NOx), affects positively the TFP for all States.

The present paper develops a competitive Ramsey economy where pollution affects both the consumption behavior and the technology (through TFP). Within this simple framework, we find that the positive effect of pollution on TFP is able to reduce the effect of pollution on the marginal utility of consumption for which a Hopf bifurcation occurs. To put it another way, we show that the effect, empirically stressed by Empora and Mamuneas (2011), makes endogenous business cycles more likely to occur.

The paper is organized in three parts: (1) presentation of the theoretical framework, (2) discussion of conditions for the occurrence of a Hopf bifurcation and (3) conclusion.

2 The model

We consider a continuous-time Ramsey economy with pollution and capital accumulation. Pollution is a by-product of industrial activities and affects simultaneously the TFP and the individual welfare by distorting the individual consumption behavior.

2.1 Technology

At time t, a representative firm produces a single output $Y(t)$. Technology is represented by a constant returns to scale production function: $Y(t) = A(t) F(K(t), L(t))$, where $A(t)$ is the TFP, $K(t)$ and $L(t)$ are the demands for capital and labor at time t.

Assumption 1 *The production function $F : \mathbb{R}_+^2 \to \mathbb{R}_+$ is C^1, homogeneous of degree one, strictly increasing and concave. Standard Inada conditions hold.*

The firm chooses the amount of capital and labor to maximize its profit taking as given the real interest rate $r(t)$ and the real wage $w(t)$. In the following, for notational simplicity, we will omit the time argument t.

The program $\max_{K,L} [F(K,L) - rK - wL]$ is correctly defined under assumption 1 and the first-order conditions write:

$$r = Af'(k) \equiv Ar(k) \tag{1}$$
$$w = A[f(k) - kf'(k)] \equiv Aw(k) \tag{2}$$

where $f(k) \equiv F(k,1)$ is the average productivity and $k = k(t) \equiv K(t)/L(t)$ denotes the capital intensity at time t. We introduce the capital share in total income α and the elasticity of capital-labor substitution σ:

$$\alpha(k) \equiv \frac{kf'(k)}{f(k)}$$
$$\sigma(k) = \alpha(k)\frac{w(k)}{kw'(k)}$$

2.2 The representative household

The household earns a capital income rh and a labor income wl where $h = h(t)$ and $l = l(t)$ denote the individual wealth and labor supply at time t. For simplicity, we assume that the household supplies inelastically his labor such that $l = 1$. The household's incomes are consumed and saved/invested according to the budget constraint:

$$\dot{h} \leq (r - \delta)h + w - c \tag{3}$$

The gross investment includes the capital depreciation at the rate δ.

For simplicity, the population of consumers-workers is constant over time: $N = 1$. Such normalization implies $L = Nl = l = 1$, $K = Nh = h$ and $h = K/N = kl = k$.

In the following, P denotes the aggregate level of air pollution.

Assumption 2 *Preferences are given by the following non-separable utility function: $u(c,P)$ with $u_c > 0$, $u_P < 0$ as first-order restrictions and $u_{cc} < 0$, $u_{cP} > 0$ as second-order restrictions, and $\lim_{c\to 0^+} u_c = \infty$ as a limit conditions.*

The condition $u_{cP} > 0$ implies that pollution enhances the marginal utility of consumption. This is the *compensation effect* stressed by Michel and Rotillon (1996).

We introduce the two following elasticities :

$$\varepsilon_{cc} \equiv \frac{cu_{cc}}{u_c}$$
$$\varepsilon_{cP} \equiv \frac{Pu_{cP}}{u_c}$$

Where ε_{cc} is the inverse of the well-known intertemporal elasticity of consumption and ε_{cP} captures the effect of pollution on marginal utility of consumption. Assumption 2 implies that $\varepsilon_{cc} < 0$ and $\varepsilon_{cP} > 0$.

The agent maximizes the intertemporal utility function $\int_0^\infty e^{-\rho t} u(c, P)\, dt$ under the budget constraint (3) where $\rho > 0$ is the rate of time preference. This program is correctly defined under assumption 2.

Proposition 1 *The first-order conditions result in a static relation*

$$u_c - \lambda = 0 \tag{4}$$

a dynamic Euler equation $\dot{\lambda} = \lambda(\rho + \delta - r)$ *and the budget constraint (3) now binding* $\dot{k} = (r - \delta)k + w - c$ *jointly with the transversality condition* $\lim_{t\to\infty} e^{-\rho t}\lambda(t)k(t) = 0$.

Proof. As usual the Hamiltonian function write :

$$H = e^{-\rho t}u(c, P) + \tilde{\lambda}[(r - \delta)k + w - c]$$

posing $\lambda = e^{\rho t}\tilde{\lambda}$ and the last proposition follows. ■

Assumption 2 ensures that the first order conditions given in proposition 1 are necessary and sufficient for optimality (See Mangasarian 1966).

2.3 Pollution

The aggregate stock of air pollution P is a pure externality. Technology is dirty and pollution persists. As in Michel and Rotillon (1996), we assume a simple linear process:

$$\dot{P} = -aP + bY \tag{5}$$

where $a \geq 0$ captures the natural rate of pollution absorption and $b \geq 0$ the environmental impact of production. According to assumption 1, the process of pollution accumulation (5) writes:

$$\dot{P} = -aP + bAf(k)$$

In addition, as it was empirically stressed by Empora and Mamuneas (2011), we assume that pollution positively affect the TFP according to:

Assumption 3 *The total factor productivity function* $A : \mathbb{R}_+ \to \mathbb{R}_+$ *is* C^1 *and strictly increasing:* $A'(P) > 0$ *for every* $P \geq 0$. *The following boundary conditions hold:* $\lim_{P\to 0} A(P) = 0$ *and* $\lim_{P\to+\infty} A(P) = +\infty$.

We introduce the following elasticity :

$$\theta \equiv \frac{PA'(P)}{A(P)}$$

θ captures the effect of pollution on the TFP.

2.4 Equilibrium

At the equilibrium, all markets clear. Applying the Implicit Function Theorem on the static relation (4), we obtain that c is a function of (λ, P), that is $c = c(\lambda, P)$ such that :

$$\frac{\lambda}{c}\frac{\partial c}{\partial \lambda} = \frac{1}{\varepsilon_{cc}}$$
$$\frac{P}{c}\frac{\partial c}{\partial P} = -\frac{\varepsilon_{cP}}{\varepsilon_{cc}}$$

It follows from assumption 2 that $\frac{\partial c}{\partial \lambda} < 0$ and $\frac{\partial c}{\partial P} > 0$.

Proposition 2 *The equilibrium transition is represented by the following dynamic system:*

$$\frac{\dot{\lambda}}{\lambda} = g_1(\lambda, k, P) = \rho + \delta - A(P)\,r(k) \tag{6}$$
$$\frac{\dot{k}}{k} = g_2(\lambda, k, P) = A(P)\,r(k) - \delta + A(P)\frac{w(k)}{k} - \frac{c(\lambda, P)}{k} \tag{7}$$
$$\frac{\dot{P}}{P} = g_3(\lambda, k, P) = -a + b\frac{A(P)}{P}f(k) \tag{8}$$

Proof. Simply consider equations (1) and (2), proposition 1 and assumption 3. ■

2.5 Steady state

Our task now is to ensure the existence of a stationary solution for the system defined by equations (6), (7) and (8) and to question its uniqueness. Since $w(k) = f(k) - r(k)\,k$, a stationary solution is a triplet $(\lambda^*, k^*, P^*) \in \mathbb{R}^3_+$ satisfying :

$$A(P)\,r(k) = \rho + \delta \tag{9}$$
$$\frac{a}{b}P - k\delta = c(\lambda, P) \tag{10}$$
$$\frac{A(P)}{P}f(k) = \frac{a}{b} \tag{11}$$

Proposition 3 *The system (6), (7),(8) may have multiple stationary solutions.*

Proof. See appendix. ■

To facilitate the analysis of the local dynamics, we assume now the following specifications :

$$f(k) = k^{\alpha} \tag{12}$$
$$A(P) = P^{\theta} \tag{13}$$
$$u(c, P) = \frac{(cP^{-\eta})^{1-\varepsilon}}{1-\varepsilon} \tag{14}$$

the next proposition gives conditions for the existence of a normalized steady state (NSS)[1]

Proposition 4 *If there is no capital depreciation ($\delta = 0$), if $\alpha = \rho$ and $a = b$, then, for every $\theta \neq 1 - \alpha$, there is a unique steady state such that $\lambda^* = k^* = P^* = 1$.*

Proof. Considering specification (12), (13) and (14), equations (9), (10) and (11) become :

$$\alpha P^{\theta} k^{\alpha-1} = \rho + \delta \tag{15}$$
$$\frac{a}{b} P - k\delta = \lambda^{-\frac{1}{\varepsilon}} P^{\eta\left(1-\frac{1}{\varepsilon}\right)} \tag{16}$$
$$P^{\theta-1} k^{\alpha} = \frac{a}{b} \tag{17}$$

since $\delta = 0$, $\alpha = \rho$ and $a = b$, equation (15) gives that $k = P^{\frac{\theta}{1-\alpha}}$, injecting this relation into (17) gives that $P^{\frac{\theta-(1-\alpha)}{1-\alpha}} = 1$. Since $\theta \neq 1-\alpha$, the steady state is unique and is such that $\lambda^* = k^* = P^* = 1$. The last proposition follows. ■

3 Conditions for the occurrence of a Hopf bifurcation

Our task now is to study the dynamics near the NSS, more precisely, we want to compare conditions for which a Hopf bifurcation occurs when $\theta = 0$ and when $\theta > 0$.

The utility function (14) implies that $\varepsilon_{cc} \equiv -\varepsilon$ and $\varepsilon_{cP} \equiv \eta(\varepsilon - 1)$. We set :

$$\beta \equiv \varepsilon_{cP} \equiv \eta(\varepsilon - 1)$$

Assumption 2 implies that $\beta > 0$.

Proposition 5 *The Jacobian matrix J, evaluated at the NSS is defined by :*

$$J = \begin{bmatrix} \frac{\partial g_1}{\partial \lambda} & \frac{\partial g_1}{\partial k} & \frac{\partial g_1}{\partial P} \\ \frac{\partial g_2}{\partial \lambda} & \frac{\partial g_2}{\partial k} & \frac{\partial g_2}{\partial P} \\ \frac{\partial g_3}{\partial \lambda} & \frac{\partial g_3}{\partial k} & \frac{\partial g_3}{\partial P} \end{bmatrix} = \begin{bmatrix} 0 & \alpha(1-\alpha) & -\alpha\theta \\ \frac{1}{\varepsilon} & \alpha & \theta - \frac{\beta}{\varepsilon} \\ 0 & \alpha a & a(\theta - 1) \end{bmatrix}$$

[1] See Cazzavillan et al (1998) among others for more details.

Proof. See appendix. ■

Let φ the characteristic polynomial :

$$\varphi(\xi) = \xi^3 - T\xi^2 + S\xi - D$$

with :

$$D = \xi_1\xi_2\xi_3 = \frac{a\alpha}{\varepsilon}(1-\alpha-\theta) \tag{18}$$

$$S = \xi_1\xi_2 + \xi_2\xi_3 + \xi_3\xi_1 = -\frac{\alpha}{\varepsilon}(1-\alpha+a(\varepsilon-\beta)) \tag{19}$$

$$T = \xi_1 + \xi_2 + \xi_3 = \alpha + a(\theta-1) \tag{20}$$

where ξ_1, ξ_2 and ξ_3 denote the three roots of φ, that is, the eigenvalues of J.

Assumption 4 *$\alpha > a$.*

Proposition 6 *J possesses two purely imaginary eigenvalues if and only if $D = ST$ with $S > 0$.*

Proof. (*Necessity*) We want to prove that $\mathrm{Re}(\xi_1) = \mathrm{Re}(\xi_2) = 0$ when $D = ST$. If $\xi_1 = iz$ and $\xi_2 = -iz$ with $z \neq 0$ and $i^2 = -1$, it appears that $D = z^2\xi_3$, $S = z^2$ and $T = \xi_3$, namely $D = ST$ and $S > 0$.

(*Sufficiency*) We want to show that $D = ST$ with $S > 0$ appears only when two eigenvalues are complex conjugate with a zero real part. $D = ST$ implies that :

$$(\xi_1 + \xi_2)(\xi_1 + \xi_3)(\xi_2 + \xi_3) = 0$$

That is, $\xi_1 = -\xi_2$, or $\xi_1 = -\xi_3$ or $\xi_2 = -\xi_3$. Without loss of generality we consider the case where $\xi_1 = -\xi_2$ and $\xi_3 \neq 0$. In such a configuration, $S > 0$ imply $\left[-(\xi_2)^2\right] > 0$. This is possible only if ξ_2 is nonreal. If ξ_2 is nonreal, then ξ_1 is conjugated and since $\xi_1 = -\xi_2$, they have a zero real part. ■

Let :

$$\beta_1 = \frac{\alpha(1-\alpha) + a\varepsilon(\alpha-a)}{a(\alpha-a)}$$

$$\beta_2 = \frac{1-\alpha}{a} + \varepsilon$$

and :

$$\theta^* = \frac{a(a-\alpha)(\beta-\varepsilon) + \alpha(1-\alpha)}{a(\alpha + a(\beta-\varepsilon))}$$

Remark 1 $\beta_1 - \beta_2 = \frac{1-\alpha}{\alpha-a} > 0$ (see assumption 4).

Proposition 7 *Assume that pollution has no effect on the TFP ($\theta = 0$) :*
If $\beta < \beta_1$, J possesses two stable eigenvalues and an unstable one.
If $\beta > \beta_1$, J possesses three unstable eigenvalues.
When $\beta = \beta_1$, a limit cycle occurs near the NSS through a Hopf bifurcation.

Proof. When $\theta = 0$, for every $\beta > 0$, $D > 0$, then there is only two possible configurations, namely: 1) $\xi_1 < 0$, $\xi_2 < 0$, $\xi_3 > 0$ or 2) $\xi_1 > 0$, $\xi_2 > 0$, $\xi_3 > 0$. If $\beta < \beta_2$, $S < 0$ and then $\xi_1 < 0$, $\xi_2 < 0$, $\xi_3 > 0$. Following proposition 6 and consider relations (18), (19) and (20), $\xi_1 = iz$ and $\xi_2 = -iz$ with $z \neq 0$ and $i^2 = -1$ when $\beta = \beta_1$, indeed $D = ST$ with $S = a\alpha\frac{1-\alpha}{\varepsilon(\alpha-a)}$ (assumption 4 ensures that $S > 0$). In addition, $\forall\beta \neq \beta_1$ we have $D \neq ST$ which implies that ξ_1 and ξ_2 cross the imaginary axis with a non zero speed and since $\forall\beta \in \mathbb{R}$, $D \neq 0$ (implying that $\xi_3 \neq 0$), it follows that a Hopf bifurcation occurs if and only if $\beta = \beta_1$ (see Hale and Koçak (1991) among others). Finally, since there is no room for a saddle-node bifurcation ($D > 0$), the fact that $\beta_1 > \beta_2$ indicates that $\xi_1 < 0$, $\xi_2 < 0$, $\xi_3 > 0$ for every $\beta < \beta_1$. Finally, at the Hopf bifurcation point, two eigenvalues change their sign simultaneously, that is, for every $\beta > \beta_1$, $\xi_1 > 0$, $\xi_2 > 0$, $\xi_3 > 0$. ■

The next proposition gives conditions for which a Hopf bifurcation occurs when pollution is assumed to enhance the TFP.

Proposition 8 *Assume that pollution enhance the TFP ($\theta > 0$) and assume also that $\beta_2 < \beta < \varepsilon + \frac{\alpha(1-\alpha)}{\alpha-a}$:*
If $\theta < \theta^*$, J possesses two stable eigenvalues and an unstable one.
If $\theta^* < \theta < 1-\alpha$, J possesses three unstable eigenvalues.
If $\theta > 1-\alpha$, J possesses two unstable eigenvalues and an stable one.
When $\theta = \theta^*$ a limit cycle occurs near the NSS through a Hopf bifurcation and when $\theta = 1-\alpha$, a saddle-node bifurcation occurs.

Proof. Assumption 4 implies that $T > 0$, that is, there is always an unstable eigenvalue (indeterminacy is ruled out). In addition, $D > 0$ when $\theta < 1-\alpha$, $D = 0$ when $\theta = 1-\alpha$ and $D < 0$ when $\theta > 1-\alpha$. It follows that a saddle-node bifurcation occurs near the NSS when $\theta = 1-\alpha$, in such a case the eigenvalues are simply given by :

$$\xi_1 = \frac{1}{2}\alpha(1-a) + \frac{1}{2}\sqrt{\frac{\alpha}{\varepsilon}\left(\alpha\varepsilon(a-1)^2 - 4\left(a(\beta-\varepsilon) - (1-\alpha)\right)\right)}$$
$$\xi_2 = \frac{1}{2}\alpha(1-a) - \frac{1}{2}\sqrt{\frac{\alpha}{\varepsilon}\left(\alpha\varepsilon(a-1)^2 - 4\left(a(\beta-\varepsilon) - (1-\alpha)\right)\right)}$$
$$\xi_3 = 0$$

the condition $\beta < \varepsilon + \frac{\alpha(1-\alpha)}{\alpha-a}$ ensures that ξ_1 and ξ_2 are two positive real number and since $D < 0$ when $\theta > 1-\alpha$, it follows that $\xi_1 > 0$, $\xi_2 > 0$ and $\xi_3 < 0$ (remember that $T > 0$) when $\theta > 1-\alpha$.

In addition $\theta = \theta^*$ implies $D = ST$ with $S > 0$ (indeed $\beta > \beta_2$), following proposition 6, it induces that $\xi_1 = iz$ and $\xi_2 = -iz$ with $z \neq 0$ and $i^2 = -1$. In addition, $\forall\theta \neq \theta^*$, we have $D \neq ST$ which implies that ξ_1 and ξ_2 cross the imaginary axis with a non zero speed and since $\theta^* \neq 1-\alpha$, $\xi_3 \neq 0$, thus a Hopf bifurcation occurs near the NSS if and only if $\theta = \theta^*$ (see Hale and Koçak (1991) among others).

Finally, for $\beta > \beta_2$:

$$(1-\alpha) - \theta^* = \frac{\alpha(1-a)}{a} \frac{\alpha + a(\beta - \varepsilon) - 1}{\alpha + a(\beta - \varepsilon)} > 0$$

Since a Hopf bifurcation indicates that two eigenvalues change their sign simultaneously, proposition 8 follows. ■

Remark 2 *The occurrence of a saddle-node bifurcation is not surprising since the NSS loses its uniqueness when $\theta = 1-\alpha$. (See proof of proposition 4).*

Proposition 9 *The positive pollution effect on the TFP ($\theta > 0$) reduces the effect of pollution on the marginal utility of consumption for which a limit cycle occurs.*

Proof. Simply consider propositions 7, 8 and remark 1. ■

4 Conclusion

Through this paper, we have developed a Ramsey model in which pollution affects positively both the TFP and the consumption behavior. After giving the general conditions for which a Hopf bifurcation occurs, we find that the positive effect of pollution on the TFP, empirically stressed by Empora and Mamuneas (2011), makes endogenous business cycles more likely to occur since it reduces the effect of pollution on the marginal utility of consumption for which a Hopf bifurcation occurs.

5 Appendix

Proof of proposition 3 :

Applying the implicit functions theorem on equation (9) gives that $k = k(P)$ with :

$$k'(P) = -\frac{A'(P)\,r(k)}{A(P)\,f''(k)} > 0$$

equations (10) then becomes :

$$\frac{a}{b}P - \delta k(P) = c(\lambda, P) \tag{21}$$

From the implicit function theorem, equation (21) gives that $\lambda = \psi(P)$ with :

$$\psi'(P) = -\frac{\frac{\partial c}{\partial P} - \frac{a}{b} + \delta k'(P)}{\frac{\partial c}{\partial \lambda}}$$

Let $\mu(P) = \frac{A(P)}{P} f(k(P))$, it follows :

$$\mu'(P) = \frac{A(P)\,f(k(P))(\theta - 1) + A(P)\,f'(k)\,k'(P)\,P}{P^2}$$

assumption 2 and 3 imply the non-monotonicity of $\psi(P)$ and $\mu(P)$, indeed $\psi'(P) \lesseqgtr 0$ and $\mu'(P) \lesseqgtr 0$, and thus the possible multiplicity of stationary solutions.

■

Evaluation of the Jacobian matrix given in proposition 5 :

Derivatives of $g_1(\lambda, k, P)$:

$$\frac{\partial g_1(\lambda, k, P)}{\partial \lambda} = 0$$
$$\frac{\partial g_1(\lambda, k, P)}{\partial k} = \frac{\lambda}{k}\frac{(\rho + \delta)(1 - \alpha)}{\sigma}$$
$$\frac{\partial g_1(\lambda, k, P)}{\partial P} = -\frac{\lambda}{P}\theta(\rho + \delta)$$

Indeed, at the steady state, $A(P)\,r(k) = \rho + \delta$

Derivatives of $g_2(\lambda, k, P)$:

$$\frac{\partial g_2(\lambda, k, P)}{\partial \lambda} = -\left(\frac{a}{b}\frac{P}{\lambda} - \delta\frac{k}{\lambda}\right)\frac{1}{\varepsilon_{cc}}$$
$$\frac{\partial g_2(\lambda, k, P)}{\partial k} = (\rho + \delta)\left[1 - \left(\frac{1 - \alpha}{\sigma}\right)\right] - \delta + \left(\frac{a}{b}\frac{P}{k} - \rho - \delta\right)\frac{\alpha}{\sigma}$$
$$\frac{\partial g_2(\lambda, k, P)}{\partial P} = \theta\frac{a}{b} + \left(\frac{a}{b} - \delta\frac{k}{P}\right)\frac{\varepsilon_{cP}}{\varepsilon_{cc}}$$

indeed, notice that, at the steady state, $A(P)\,w(k) = \frac{a}{b}P - (\rho + \delta)\,k$ and $\frac{c}{P} = \frac{a}{b} - \delta\frac{k}{P}$.

Derivatives of $g_3(\lambda, k, P)$:

$$\frac{\partial g_3(\lambda, k, P)}{\partial \lambda} = 0$$
$$\frac{\partial g_3(\lambda, k, P)}{\partial k} = b(\rho + \delta)$$
$$\frac{\partial g_3(\lambda, k, P)}{\partial P} = a(\theta - 1)$$

Indeed, at the steady state, $\frac{a}{b} = \frac{A(P)f(k)}{P}$.
The Jacobian matrix becomes :

$$J = \begin{bmatrix} 0 & \frac{\lambda}{k}\frac{(\rho+\delta)(1-\alpha)}{\sigma} & -\frac{\lambda}{P}\theta(\rho + \delta) \\ -\left(\frac{a}{b}\frac{P}{\lambda} - \delta\frac{k}{\lambda}\right)\frac{1}{\varepsilon_{cc}} & B & \theta\frac{a}{b} + \left(\frac{a}{b} - \delta\frac{k}{P}\right)\frac{\varepsilon_{cP}}{\varepsilon_{cc}} \\ 0 & b(\rho + \delta) & a(\theta - 1) \end{bmatrix}$$

With $B = (\rho + \delta)\left(1 - \left(\frac{1-\alpha}{\sigma}\right)\right) - \delta + \left(\frac{a}{b}\frac{P}{k} - \rho - \delta\right)\frac{\alpha}{\sigma}$
At the NSS, $a = b$, $\delta = 0$, $\rho = \alpha$, $\lambda = k = P = 1$ and $\sigma = 1$ (because $f(k) = k^{\alpha}$).
That is why, at the NSS, J becomes simply :

$$J = \begin{bmatrix} 0 & \alpha(1 - \alpha) & -\alpha\theta \\ \frac{1}{\varepsilon} & \alpha & \theta - \frac{\beta}{\varepsilon} \\ 0 & \alpha a & a(\theta - 1) \end{bmatrix}$$

■

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