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Pareto Distribution of Firm Size and Knowledge Spillover Process as a Network

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Abstract

The firm size distribution is considered as Pareto distribution. In the present paper, we show that the Pareto distribution of firm size results from the spillover network model which was introduced in Konno (2010).

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1. Introduction

It has been studied that the firm size distribution follows Pareto distribution, which means that $P(s) \sim s^{-x}$ where s denotes the firm size, $P(s)$ denotes the distribution function of firm size s, and x is some constant. If $x = 2$, the distribution is called Zipf's law. It has been studied that the constant x is approximately equal to 2 so far. In particular, it has been shown that the tail of the firm size distribution follows Pareto distribution well. The reason why firm distribution follows Pareto distribution has been studied so far Gabaix (2009). In the present paper, we show that the Pareto distribution of firm size is derived from the model of knowledge spillover process as a network which was introduced in Konno (2010). If we assume that the network is a scale-free one, the firm size distribution is Pareto one. We also show that a scale-free network is constructed from the likely mechanism based on the spillover network model. A scale-free network is ubiquitously observed in reality. Degree is a number of links a vertex has. Heterogeneity of network is the magnitude of variance of its degree distribution. One characteristics of networks in reality is their large heterogeneity. If network heterogeneity is large, the network has small number of hubs which have a lot of links and large number of vertices which have only small number of links. A representative of heterogeneous network is a scale-free one. For that reason, a scale-free network has been studied a lot in models on networks and in network formation mechanism. A representative of homogeneous network is a regular one. The mechanism which results in Pareto firm size distribution has been attract much attention. A classical paper Simon and Bonini (1958) studied stochastic growth mechanism of firms and derive the Pareto distribution. Axtell (2001) showed that Zipf's law characterizes firm sizes Using data on the entire population of tax-paying firms in the United States. Reed (2001) derived the double Pareto distribution in the following way, where double Pareto distribution is the power-law distribution which has different exponents in upper tail and lower tail. Suppose the evolution of firm size follows geometric Brownian motion (GBM) and also suppose that the time T when the firm size is observed is also random, then the firm size distribution follows double Pareto distribution. Luttmer (2007) studied the model where growth is the result of idiosyncratic firm productivity improvements, selection of successful firms, and imitation by entrants. They showed that Zipf's law can be interpreted to mean that entry costs are high or that imitation is difficult, or both.

2. Spillover Network and Firm Size Pareto Distribution

2.1. Spillover on Networks

Figure 1 illustrates the knowledge spillover on networks. Although we wrote knowledge spillover, we regard that what actually spillovers is Total factor productivity (TFP). Hence, what actually spillovers is unknown, since what is TFP itself is not fully understood yet. The vertices represent firms, although they can represent countries, cities, or other things depending on the situation one studies. The network represents the spillover relationship. A firm receives the spillover directly form adjacent firms in one time step. For instance, firm A receives the spillover from adjacent firms, C, B, D, and E. A firm receives the

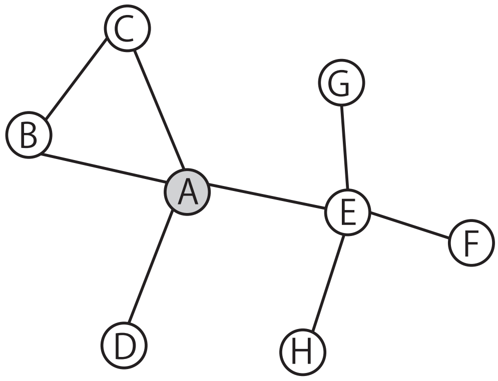

Figure 1: Firms on a knowledge spillover network

spillover from unconnected firms through connected firms after a while. For example, firm A receives spillover from firm F through firm B in two time steps.

Let A_j denote the TFP of firm j, ξ_j denote the degree of firm j where degree is the number of links the vertex has, $A_j(\xi_j, t)$ denote the TFP of firm j with degree xi_j at time t, ρ denote the growth rate of TFP without spillover, w_{ji} denote the weight of link between firm i and firm j which specify the strength of spillover between these two firms. If the weight w_{ji} is large the spillover between firm i and firm j is large. However, we suppose that all the weights between any firms are the same weight w for simplicity. Since we focus on spillover through networks, we assume growth rate ρ without spillover constant. The spillover process on network is studied in Konno (2010) and the equation which describes the spillover on networks is given by

$$A_j(\xi_j, t + \triangle t) = (1 + \rho \triangle t) A_j(\xi_j, t) + \sum_{i \in \partial j} w A_i(\xi_i, t) \triangle t, \tag{1}$$

where the $i \in \partial j$ denote all the firms adjacent to firm j. In Fig. 1, $\partial A = \{B, C, D, E\}$ holds true. The mean-field approximate asymptotic solution as $t \to \infty$ is given by

$$A(\xi, t) = \xi \exp\left[(\rho + w\langle \xi_{\rm nn} \rangle)\, t\right]. \tag{2}$$

The numerical simulation in Konno (2010) confirms the approximate asymptotic solution.

2.2. Network Formation

This subsection is to show that a scale-free network is constructed from the likely mechanism in our spillover on networks model. A scale-free network is a network where degree distribution $P(\xi)$ follows $P(\xi) \sim \xi^{-\gamma}$. The degree distribution is the probability that a randomly chosen vertex has degree ξ. The Pareto distribution results from a scale-free network of the spillover. If you are satisfied with that scale-free network of the spillover is simply assumed, then this subsection may not be necessary. We think that to simply assume the scale-free network is one way. However, we will show that a scale-free network is constructed from a likely network generating mechanism in our spillover on networks model.

A firm receives large amount of spillover if the firm links to the firm with large amount of TFP in the spillover process described in eq. 1. Therefore, a firm likes to link to the firm with large amount of TFP in the model. This tendency is described by the following network formation process. At each time step, one new firm enters the existing spillover network and links to m existing firms with some probability. Because of the above observation that a firm likes to link to a high-TFP firm, we introduce such a stochastic mechanism that the probability $\Pr(A_j)$ an existing firm with A_j unit of TFP attracts a new link is given by

$$\Pr(A_j) = \frac{A_j}{\sum_i A_i}. \tag{3}$$

In the following time steps, this process continues. This is a stochastic network formation mechanism. We let $p(A(\xi), s, t)$ denote the probability that the firm which entered the network at time s has $A(\xi)$ unit TFP at time t. The process is described by the following master equation:

$$p(A(\xi), s, t+1) = \frac{mA(\xi-1)}{\sum_i A(\xi_i)} p(A(\xi-1), s, t) + \left(1 - \frac{mA(\xi)}{\sum_i A(\xi_i)}\right) p(A(\xi), s, t), \tag{4}$$

because the firm with TFP $A(\xi-1)$ increases the degree by 1 with probability $m \cdot \Pr(A(\xi-1))$. We investigate asymptotic degree distribution as time t approaches infinity. We let $p(A)$ denote the distribution of TFP A, which is given by

$$p(A) = \lim_{t\to\infty} \sum_{s=1}^{t} \frac{p(A, s, t)}{t}. \tag{5}$$

We take the summation, $\sum_{s=1}^{t+1}$, of both sides of Eq. (4) with noting $p(A, t+1, t) = 0$, because at time t the firm that enters at time $t+1$ does not exist by definition. Thus, we have

$$\begin{aligned}\sum_{s=1}^{t+1} p(A(\xi), s, t+1) =& \frac{mA(\xi-1)}{\sum_i A(\xi_i)} \sum_{s=1}^{t} p(A(\xi-1), s, t) \\ &+ \left(1 - \frac{mA(\xi)}{\sum_i A(\xi_i)}\right) \sum_{s=1}^{t} p(A(\xi), s, t).\end{aligned} \tag{6}$$

Substituting Eq. (5) into the above Eq. (6) to find asymptotic distribution, we obtain

$$(t+1)p(A(\xi)) = \frac{mA(\xi-1)}{\sum_i A(\xi_i)} tp(A(\xi-1)) + \left(1 - \frac{mA(\xi)}{\sum_i A(\xi_i)}\right) tp(A(\xi)). \tag{7}$$

Notice that the mean-field asymptotic solution of spillover equation is given by

$$A(\xi) = \xi \exp\left[(\rho + w\langle\xi_{\rm nn}\rangle)t\right]. \tag{8}$$

We now assume that sufficient time elapses between two firms enter the network, so that Eq (8) holds. Consequently, the stochastic dynamics becomes that of Barabasi and Albert (1999). Thus, we will make use of it. At each time step m links are added to the network. If we sum up the degrees of all the vertices in the network at time t, we obtain $2mt$ edges because every single link is counted twice. Thus, $\sum_i \xi_i \sim 2mt$ holds. We have the following probability using Eq. (8):

$$\begin{aligned} m\,\Pr(A(\xi-1)) &= \frac{mA(\xi-1)}{\sum_i A(\xi_i)} \\ &= \frac{m(\xi-1)\exp\left[(\rho+\delta_N w\langle\xi_{\rm nn}\rangle)t\right]}{\sum_i \xi_i \exp\left[(\rho+\delta_N w\langle\xi_{\rm nn}\rangle)t\right]} \\ &= \frac{\xi-1}{2t}. \end{aligned} \tag{9}$$

Substituting Eq. (9) into Eq. (7), we find

$$(t+1)p(A(\xi)) = \frac{\xi-1}{2t}tp(A(\xi-1) + \left(1-\frac{\xi}{2t}\right)tp(A(\xi)) \tag{10}$$

which transforms

$$p(A(\xi)) = \frac{\xi-1}{\xi+2}p(A(\xi-1)). \tag{11}$$

After solving this equation, we get

$$p(A(\xi)) = \frac{\rm Const}{\xi(\xi+1)(\xi+2)} \sim \xi^{-3}, \tag{12}$$

which tells that the network generated by this stochastic mechanism is the scale-free network with the exponent $\gamma = 3$. We now comment on the fact that the network generating mechanism is stochastic one in that which firm to link is determined stochastically. If the network formation mechanism is deterministic in that which firm to link is not stochastic, then the resulting network is likely star network in which only one hub firm links to all the other firms which have only one link. The existing study has shown that heterogeneous network like scale-free one is ubiquitous but star network is not. A heterogeneous network is a network whose variance of the degree distribution is large. For example, a network of co-authorship between investors is not a star one but a heterogeneous one. In order to make a heterogeneous network which is often observed in reality by simple mechanism, it seems that we need to have a stochastic network formation network. The fact that a heterogenous network such as scale-free one is often observed in reality is the reason why scale-free networks have been studied in views of network formation mechanism and models on networks. The exponent γ of scale-free networks in reality typically satisfies $2 < \gamma \leq 3$. Network formation models of similar preferential attachment mechanisms with various exponents γ have been studied.

2.3. Monopolistic Firms on Networks and the Consumer

We assume the following economy. The utility function U of consumer is given by

$$U = \left(\int_1^N X_j^{\frac{\sigma-1}{\sigma}} dj \right)^{\frac{\sigma}{\sigma-1}}, \tag{13}$$

where X_j is jth final goods produced by monopoly firm j, there are N final goods produced by each monopoly firm, and $\sigma \in [0, \infty)$. The firms are on the spillover network. The ξ_j is the degree of firm j. One firm produces one kind of final goods. The network is a scale-free one which is constructed by the mechanism explained in Section 2.2. Or we can simply assume that the network is a scale-free one. The budget constraint for the consumer is given by.

$$\int_1^N p_j X_j \, dj = I, \tag{14}$$

where household income is denoted by I and the price of the jth goods X_j is denoted by p_j. The demand function for final goods $X_j(p_j)$ is given by

$$X_j(p_j) = \frac{p_j^{-\sigma}}{\int_1^N p_i^{1-\sigma} di} I = p_j^{-\sigma} \left(\frac{I}{P} \right), \tag{15}$$

where $P \equiv \int_1^N p_j^{1-\sigma} dj$.

The technology of monopolistic firm j is to change one unit of labor into A_j unit jth final goods; namely, $X_j = A_j l_j$, where l_j is the labor employed by firm j. The profit maximization problem of firm j is given by

$$\max_{p_j} \; p_j X_j(p_j) - w l_j, \tag{16}$$

where wage is denoted by w. Substituting Eq. (15) into Eq. (16), we have the price p_j of jth final goods:

$$p_j = \frac{\sigma}{\sigma - 1} w A_j^{-1}. \tag{17}$$

The household supplies L unit labor inelastically. Thus, the labor market clearing condition is given by

$$\int_1^N l_j dj = L. \tag{18}$$

Combining above equations together, we have,

$$X_j = \frac{A_j^{\sigma}}{\int_1^N A_i^{\sigma-1} di} L. \tag{19}$$

Finally, we have the size of firm j given by

$$p_j X_j = \frac{\sigma}{\sigma - 1} \frac{A_j^{\sigma-1}}{\int_1^N A_i^{\sigma-1} di} wL. \tag{20}$$

2.4. The Pareto Distribution of Firm size

Remember that the asymptotic mean-field solution of spillover equation is given by

$$A(\xi, t) = \xi \exp\left[\left(\rho + w\langle\xi_{\mathrm{nn}}\rangle\right) t\right] \tag{21}$$

and that the network is a scale-free network. The scale-free network is constructed by the mechanism in Section 2.2 or that the network is a scale-free one is assumed. Because the TFP of a firm with degree ξ is given by eq. (21), the TFP of the firm is proportional to the degree of the firm. Since the degree distribution of a scale free network is $P(\xi) \sim \xi^{-\gamma}$, the distribution function $P(A)$ of TFP follows Pareto distribution $P(A) \sim A^{-\gamma}$. We use $P(\bullet)$ for different distribution functions if we do not see any confusion. Let s denote the size pX of firm. The size distribution function $P(s)$ of firm is given by

$$\Pr(s) \sim s^{\frac{-(\sigma-2+\gamma)}{\sigma-1}}, \tag{22}$$

which is a Pareto distribution as desired. The size distribution in terms of employee l_j has the same distribution.

3. Conclusion

The firm size distribution is considered as Pareto distribution, which means that $P(s) \sim s^{-x}$, where s denotes the firm size and x is some constant. In the present paper, we show that the Pareto distribution of firm size is derived from the spillover network model which was introduced in Konno (2010).

Axtell, R. L., 2001. Zipf distribution of us firm sizes. Science 293 (5536), 1818–1820.

Barabasi, A.-L., Albert, R., 1999. Emergence of scaling in random networks. Science 286 (5439), 509–512.

Gabaix, X., 2009. Power laws in economics and finance. Annual Review of Economics 1 (1), 255–294.

Konno, T., 2010. Knowledge spillover on complex networks. Rieti Discussion Paper Series, 10–E–02.

Luttmer, E. G., 2007. Selection, growth, and the size distribution of firms. The Quarterly Journal of Economics 122 (3), 1103–1144.

Reed, W. J., 2001. The pareto, zipf and other power laws. Economics Letters 74 (1), 15–19.

Simon, H. A., Bonini, C. P., 1958. The size distribution of business firms. The American Economic Review 48 (4), 607–617.