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How to Add Apples and Pears: Non-Symmetric Nash Bargaining and the Generalized Joint Surplus

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Abstract

We find how to compute the non-symmetric Nash bargaining solution by means of a generalized property of linear division of the joint surplus, as an alternative of solving the maximization of the generalized Nash product. This generalized property of linear division in the non-symmetric Nash bargaining solution can be applied to the case when bargainers use different utility scales, in particular when they have different attitudes toward risk, as in the case of a risk neutral firm and a risk averse individual. The surplus each agent receives has to be expressed in compatible, or comparable, units across agents. This is contrary to what has been believed in the labor literature, where many authors have partially expressed surpluses in comparable units. We finally illustrate the conditions of applicability of our result by means of a well-known example.

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1 Introduction

In order to compute the non-symmetric Nash bargaining solution there is a property that is well-known to, and widely-used by, economists.[1] When utilities of agents are perfectly transferable, the surplus from bargaining obtained by each individual at the non-symmetric Nash bargaining solution coincides with a linear sharing of the total surplus. In other words, the weight assigned to an individual in the generalized Nash product is the proportion of the total surplus that she receives at the corresponding non-symmetric Nash bargaining solution. This *linear equivalence* allows us to choose between maximizing the generalized Nash product or finding the unique solution that linearly shares the joint surplus. Using the linear sharing rule correctly can be easier, and faster, in computational work.

The purpose of this paper is to show how to compute the non-symmetric Nash bargaining solution by means of a linear sharing of the total surplus from bargaining, instead of maximizing the generalized Nash bargaining product, when utilities are not perfectly transferable, and when bargaining takes place over multiple issues.[2] To do so one has to to transform the units of the individual surpluses in proper fashion. It is useful to stress that such unit transformation is not necessary when maximizing the generalized Nash product. In addition, the appropriate unit transformation should be used when comparing quantitatively the surplus of economies with different utility parametrization. We conclude that differentiability of the Pareto frontier or utility possibility frontier has to be checked in order to correctly use our formula.

[1]See Pissarides (1985), Blanchard and Diamond (1990), Cahuc and Lehmann (2000), Frediksson and Holmlund (2001), Pissarides (2003), Cahuc and Zylberberg (2004), ch.7 and 9, Boone *et al* (2007) and Lehmann and van der Linden (2007). See among others Rogerson *et al.* (2005) for bargaining in matching models, and Rupert *et. al.* (2001) for bargaining in monetary economics. See also Manning (1987) for an example of a two-step bargaining.

[2]In all references cited above except the book by Cahuc and Zylberberg (2004) bargaining takes place over one issue. For example, in a context of labor markets, the firm and the worker bargain bilaterally about their wage. Here, we allow bargaining to take place over multiple issues, as the firm and the worker could bargain over wages, hours of work, and other benefits as extra health insurance, use of firm appliances (car, cell phones, hardware, software, individual office space), etc.

2 Setting and Results

Bargaining takes place in terms of a vector of variables $x \in \Re^m$, where $m \geq 1$ is the number of issues that are relevant in the bargaining process. The two individuals, named A and B, who are engaged in bargaining, have utilities $a(x)$ and $b(x)$, respectively, for each possible choice of x. Both $a(x)$ and $b(x)$ are twice differentiable (therefore continuous) real-valued functions in $\Re^m$. Let $a_i(x)$ and $b_i(x)$ denote the first-order partial derivatives of $a(x)$ and $b(x)$, respectively, with respect to the issue i at agreement x. We also assume that if bargaining does not succeed, agents A and B obtain disagreement payoffs denoted by d_A and d_B, both real valued.

The following additional assumptions are needed to have a well-defined bargaining problem. First, there is at least one x in $\Re^m$ such that both $a(x) > d_A$ and $b(x) > d_B$. This assumption guarantees essentiality of our bargaining problem. Second, the functions $a(x)$ and $b(x)$ are quasi-concave in $\Re^m$, with at least one of them being strictly quasi-concave (but not necessarily both of them). And third, for any $x \in \Re^m$ satisfying that $a(x) > d_A$ and $b(x) > d_B$ we have the following conditions on partial derivatives:

- Both $a_i(x)$ and $b_i(x)$, for any i, are bounded, not equal to 0, and either positive or negative for all x.

- For any $i = 1, ..., m$: $a_i(x)$ is positive for all x if and only if $b_i(x)$ is negative for all x.

In words, these conditions on partial derivatives indicate that (i) there is no satiation point for each agent with respect to each of the issues, and (ii) there is *conflict* when trying to reach an agreement for each of the issues. These are the cases where analyzing the result of bargaining is of interest. Note that if both derivatives $a_i(x)$ and $b_i(x)$ are positive for all x, agents should be clever enough to agree on the highest value possible for such an issue i, and, equivalently, if both derivatives $a_i(x)$ and $b_i(x)$ are negative, agents should be clever enough to agree on the smallest value possible for such an issue i. We can now define the Pareto set associated to these utility functions $a(x)$ and $b(x)$ as

$$F = \left\{(u, f(u)) \in \Re^2\right\}, \tag{1}$$

where $u \in a(\Re^m)$ and $f(u)$ is the value function associated to the optimization problem[3]

$$\begin{aligned} &\max_{x\in\Re^m} b(x) \\ &\text{subject to} \quad a(x) \geq u. \end{aligned} \tag{2}$$

Both functions a and b are quasi-concave, and at least one of them is strictly quasi-concave, guaranteeing the solution to be unique. Since there is conflict for any issue i at any x, the constraint in the maximization problem (2) is binding, for any value of u, at the optimal solution.[4]

We can now write the two-person bargaining problem as a pair (S, d), where the set of feasible utilities S is defined as:

$$S \equiv \left\{(u,v) \in \Re^2 \text{ such that } \exists x \in \Re^m \text{ with } u \leq a(x) \text{ and } v \leq b(x))\right\}, \tag{3}$$

and $d = (d_A, d_B)$, the disagreement payoff, is an interior point in S. The set of feasible utilities S can alternatively be defined by making use of the value function f defined above as

$$S \equiv \left\{(u,v) \in \Re^2 \text{ such that } v \leq f(u)\right\}, \tag{4}$$

Hence, the set S satisfies free-disposal by definition. Let us check that S satisfies the usual assumptions in the bargaining theory literature, namely that it is closed, upper-bounded, and convex. By the Theorem of the Maximum $f(u)$, the value function associated to the constraint maximization problem in (2), is continuous, and therefore the set S of feasible utilities is closed.[5] Since the constraint in the maximization problem (2) is binding

[3]Alternatively, we could define the maximization problem

$$\begin{aligned} &\max_{x\in\Re^m} a(x) \\ &\text{subject to} \quad b(x) \geq v, \end{aligned}$$

where $v \in b(\Re^m)$ and the utility pairs would be written as $(g(v), v)$, where $g(v)$ is the maximum value function of the problem written in terms of v.

[4]Suppose, by contradiction, that there is a value of $u \in a(\Re^m)$ such that $x^*(u)$, the solution of the problem in (2), satisfies that $a(x^*(u)) > u$. Then we can find another element $\tilde{x}$ in a small neighborhood of $x^*(u)$ differing from the latter only on the coordinates for which the corresponding partial derivative of the a function are decreasing and such that, by continuity of a, $a(x^*(u)) > a(\tilde{x}) \geq u$. Given that partial derivatives of a and b are always of opposite sign and different from zero, $b(x^*(u)) < b(\tilde{x})$. Therefore, $x^*(u)$ cannot be a solution to the problem in (2), a contradiction.

[5]See Mas Colell et al (1995), Theorem M.K.6, p. 963.

(thanks to partial derivatives being of opposite signs), the function $f(u)$ is decreasing, which guarantees that S is upper-bounded.[6] Finally, convexity of S is granted if the function $f(u)$ is concave in $u \in a(\Re^m)$, given that $f(u)$ is decreasing and S satisfies free disposal by definition.[7]

For generic two-person bargaining problems (S, d), the non-symmetric Nash bargaining solution (Nash 1950, 1953, and Kalai 1977) with weights $\alpha \in [0, 1]$ and $1 - \alpha$ solves

$$\max_{(u,v)\in S} (u - d_A)^{\alpha} (v - d_B)^{1-\alpha}, \tag{5}$$

where $\alpha \in (0, 1)$. Note that $\alpha \in [0, 1]$ is usually interpreted as the parameter measuring player 1's bargaining power, and when $\alpha = \frac{1}{2}$ we obtain the Nash bargaining solution originally defined and axiomatically characterized by Nash (Nash 1950, 1953).[8]

Proposition 1 *Consider the two-person bargaining problem defined just above and assume that the function f is differentiable in $a(\Re^m)$. An agreement $x^* = (x_i^*)_{i\in 1,\dots,m} \in \Re^m$ is the agreement at the non-symmetric Nash bargaining solution $(a(x^*), b(x^*))$ with $a(x^*) > d_A$ and $b(x^*) > d_B$ if and only if it satisfies*

$$a(x^*) - d_A = \alpha \left[a(x^*) - d_A - \frac{a_i(x^*)}{b_i(x^*)} (b(x^*) - d_B) \right], \tag{6}$$

[6]By contradiction suppose that there exist u and $\tilde{u}$ both in $a(\Re^m)$ such that $u > \tilde{u}$ and $f(u) \geq f(\tilde{u})$. Let $x(u)$ and $x(\tilde{u})$ denote the solutions of the maximization problem in (2) for the constraint $a(x) \geq u$ and for the constraint $a(x) \geq \tilde{u}$, respectively. Recall that, given that partial derivatives are of opposite signs, both constraints are binding at each maximization problem. This indicates that $a(x(u)) = u > \tilde{u}$, so that $x(u)$ satisfies the constraint for the maximization problem defined by $\tilde{u}$. By definition of the f function, $f(u) \geq f(\tilde{u})$ indicates that $b(x(u)) \geq b(x(\tilde{u}))$, a contradiction given that $x(\tilde{u})$ is the unique solution to (2) for the binding constraint $a(x) \geq \tilde{u}$.

[7]Let us take two elements (u, v) and $(\tilde{u}, \tilde{v})$ both in S. Take any convex combination of the two, $t(u, v) + (1 - t)(\tilde{u}, \tilde{v})$, for any $t \in [0, 1]$. By definition of S, $v \leq f(u)$ and $\tilde{v} \leq f(\tilde{u})$, which means that $tv + (1 - t)\tilde{v} \leq tf(u) + (1 - t)f(\tilde{u})$. Since f is concave, $tf(u) + (1 - t)f(\tilde{u}) \leq f(tu + (1 - t)\tilde{u})$, for any $t \in [0, 1]$, and therefore $tv + (1 - t)\tilde{v} \leq f(tu + (1 - t)\tilde{u})$, indicating, by definition of S, that $t(u, v) + (1 - t)(\tilde{u}, \tilde{v})$ also belongs to S for any $t \in [0, 1]$. Hence, S is a convex set.

[8]The product $(u - d_A)^{\alpha} (v - d_B)^{1-\alpha}$ is usually called the generalized Nash product.

and

$$b(x^*) - d_B = (1-\alpha)\left[b(x^*) - d_B - \frac{b_i(x^*)}{a_i(x^*)}\left(a(x^*) - d_A\right)\right], \tag{7}$$

for any issue i.

At the margin and in terms of exchange (or in terms of opportunity costs), one util of B is equivalent to $\left|\frac{a_i(x)}{b_i(x)}\right|$ utils of A when issue i at agreement x is taken as a reference. A obtains an α share of the total surplus when the latter is *all of it* measured in utils of A. The joint surplus measured in utils of A may differ if we fix another issue, say j, as a reference, as long as

$$\frac{a_i(x)}{b_i(x)} \neq \frac{a_j(x)}{b_j(x)}. \tag{8}$$

From (2), however, any x that generates a pair of utilities $(a(x), b(x))$ belonging to the frontier F satisfies

$$\frac{a_i(x)}{b_i(x)} = \frac{a_j(x)}{b_j(x)}.$$

Since the nonsymmetric Nash bargaining solution $(u^*, f(u^*))$ lies in the frontier F of the feasible set S, it must be that the joint surplus measured in utils of A is the same independently of the issue that we take as a reference at the nonsymmetric Nash bargaining solution. The same can be said about the joint surplus measured in utils of B. The proof of Proposition 1 is based on this fact and on the FOC for the maximization of the generalized Nash product, and it can be found in the mathematical web appendix.

From Proposition 1 we can also conclude that the agreement $x^* \in X$ at the nonsymmetric Nash bargaining solution is the unique agreement satisfying that, for every $i = 1, ..., m$

$$\frac{a(x^*) - d_A}{|a_i(x^*)|} = \alpha\left(\frac{a(x^*) - d_A}{|a_i(x^*)|} + \frac{b(x^*) - d_B}{|b_i(x^*)|}\right) \tag{9}$$

It is worth noting that, when computing the linear equivalence with respect to either of the issues, we also have to correct the surplus of the B agent, *even when his/her utility is linear.*

3 Final Comments and Conclusions

We have shown that maximizing the generalized Nash Bargaining product takes the form of a linear split of the joint surplus even when agents utilities are not perfectly transferable, after correcting individual surpluses **at the margin**. We have identified the required conditions for the bargaining problem to be well defined and for the linear sharing equivalence to hold. In terms of curvature of the utility functions, the concavity of the utility functions (but not necessarily strict concavity) is sufficient when bargaining takes place over multiple issues, as it implies convexity of the feasible set S, which in turn implies concavity of the generalized Nash product as a function of u (See Lemmas 1 and 2 of Mathermatical Appendix.) Relaxing the assumption of convexity of S is nevertheless possible.[9]

Special care has to be taken regarding the assumption of opposite signs of marginal utilities. The existence of corner solutions to the maximization problem in (2), in addition, affects the differentiability of the frontier $f(u)$, which is required for equations (6), (7), and (9) to hold. For example, let us consider the firm-union bargaining over wage and employment as in McDonald and Solow (1981), also detailed in Cahuc and Zylbeberg (2004), chapter 7, part 3.2. The firm has a profit function equal to $R(L) - wL$, where w denotes wage per worker and L denotes number of employed workers. The union's utility function is given by $L[U(w) - U(\bar{w})]$, where $\bar{w}$ denotes benefits if worker is unemployed, and U is each union member's utility function. Bargaining takes place in terms of $w \geq 0$ and $L \leq N$. As the union's utility function is increasing in both arguments w and L we need the firm's profit function to be decreasing in both arguments too so that there is conflict. We obtain that agreements can only take place if the wage is higher than the marginal revenue of labor, $w > R'(L)$. Computing equation (9) with respect to L we obtain that at the non-symmetric Nash bargaining solution (w^*, L^*) $w^* = \alpha \frac{R(L)}{L} + (1 - \alpha)R'(L)$, implying that the optimal wage

[9] The set S of feasible utilities is usually assumed to be convex since Nash 1950. The usual justifications for the use of the non-symmetric Nash bargaining solution, Kalai (1977) –using a replica argument– and Rubinstein (1982) –from a strategic point of view–, assume a convex set S. Nevertheless, some papers have dealt with the symmetric Nash bargaining solution for non convex sets S. See, among others, Roth (1977), Kaneko (1980), Maschler et al (1988), Herrero (1989), Conley and Wilkie (1991 and 1996), Zhou (1997) and Serrano and Shimomura (1998).

is equal to the weighted average of the marginal and average revenue of labor. When the function $R(L)$ is concave its marginal value is lower than its average value and hence $w^* > R'(L^*)$. Unfortunately, when $R(L)$ is a linear function the equation above collapses into $w^* = \frac{R(L^*)}{L^*} = R'(L^*)$. Such a (w^*, L^*) minimizes the generalized Nash product, because the value of the generalized Nash product would be equal to zero.[10] The problem is still essential as long as the average product is higher than the unemployment benefits.[11] In order to use the non symmetric Nash bargaining solution to solve the firm union bargaining for this case we need to proceed in two stages. If the average revenue product, being a constant, is greater than the optimal wage w we know that both utility functions are increasing in the value of L. Hence, the maximization of the generalized Nash product happens at $L^* = N$. Now we can use equation (9) to compute the optimal wage w^*,

$$\frac{U(w^*) - U(\bar{w})}{U'(w^*)} = \alpha \left[\frac{U(w^*) - U(\bar{w})}{U'(w^*)} + \frac{R(N)}{N} - w^* \right],$$

which has a solution w^* in $\left(\bar{w}, \frac{R(N)}{N}\right)$. If U were a linear function in w, w^* would be equal to $\alpha \frac{R(N)}{N} + (1-\alpha)\bar{w} < \frac{R(N)}{N} = R'(N)$. Note that the problem does not come from linearity per se, but from the fact that, under linear technology, optimal labor hours is a corner solution to the maximization problem in (2).[12] As mentioned before, the differentiability of the f function defining the utility possibility frontier is the key, because in the presence of corner solution we lose smoothness, but not continuity, in the solution function.[13] This problem might arise when bargaining takes place over multiple issues, but not when bargaining takes place over one issue, where the maximization problem to obtain the Pareto set is trivial as long as

[10] L cannot take an infinite value, but at most the value of N which is finite.

[11] We can find a w and an L such that both utility functions are strictly positive. For example, $L = N$ and $w = \frac{1}{2}(\frac{R(N)}{N} + \bar{w})$.

[12] On the contrary, if it is the union's utility function that is linear, and not the firm's one, the Nash bargaining solution also yields a s wage $w^* = \alpha \frac{R(L^*)}{L^*} + (1-\alpha)R'(L^*)$, but L^* has to satisfy $R'(L^*) = \bar{w}$, which in turn means that $w^* = \alpha \frac{R(L^*)}{L^*} + (1-\alpha)\bar{w}$.

[13] When $R(L)$ is linear but $U(w)$ is not we obtain the following value function f:

$$f(u) = \begin{cases} N\left(\frac{R(N)}{N} - U^{-1}\left(U(\bar{w}) + \frac{u}{N}\right)\right), & \text{if } U\left(\frac{R(N)}{N}\right) - U(\bar{w}) > \frac{u}{N}, \\ 0 & \text{otherwise.} \end{cases}$$

marginal utilities are of opposite sign.

Finally, we would like to stress that exploring the possibility of interpersonal comparisons of utility is beyond our intention. Binmore (2009a et 2009b) has studied interpersonal comparisons of utility from a more philosophical point of view. Our intention here is less ambitious, namely, identifying how utility units are implicitly compared at the nonsymmetric Nash bargaining solution if we were to interpret such a solution as an α, $(1-\alpha)$ division of a common pool. We do not want to, or even claim that we should, explicitly convert utility units from one individual into the other one.

References

[1] Blanchard, O. and P. Diamond (1994) "Ranking, Unemployment Duration, and Wages." *Review of Economic Studies* **61**, 417-434.

[2] Binmore, K. (2009a) "Interpersonal Comparison of Utility" in Harold Kincaid and Don Ross (eds.) *The Oxford Handbook of Philosophy of Economic Science.* Oxford University Press: Oxford, 540-559.

[3] Binmore, K. (2009b) "Interpersonal Comparison in Egalitarian Societies." *ELSE Working paper 333* Center for Economic Learning and Social Evolution. University College London.

[4] Boone, J., P. Fredikson, B. Holmlund and J.C. van Ours (2007) "Optimal Unemployment Insurance with Monitoring and Sanctions." *The Economic Journal* **117**, 399-42.

[5] Cahuc, P. and E. Lehmann (2000) "Should Unemployment Benefits Decrease with the Unemployment Spell?" *The Journal of Public Economics* **77**, 135-153 .

[6] Cahuc, P. and A. Zylberberg (2004) *Labor Economics.* MIT Press: Cambridge, Massachusetts.

But when $U(w)$ is linear but $R(L)$ is not we obtain the following value function f:

$$f(u) = R(L^*) - \bar{w}L^* - \frac{u}{L^*U'(L^*)}, \tag{10}$$

where L^* verifies $R'(L^*) = \bar{w}$.

[7] Conley, J.P. and S. Wilkie (1991) "The Bargaining Problem without Convexity." *Economics Letters* **36**, 365-369.

[8] Conley, J.P. and S. Wilkie (1996) "An Extension of the Nash Bargaining Solution to Nonconvex Problems." *Games and Economic Behavior* **13**, 26-38.

[9] Danthine, S. and N. Navarro (2010) "How to Add Apples and Pears: Non-Symmetric Nash Bargaining and the Generalized Joint Surplus." *Malaga Economic Theory Research Center Working Papers,* WP 2010-4.

[10] Fredikson, P. and B. Holmlund (2001) "Optimal Unemployment Insurance in Search Equilibrium." *The Journal of Labor Economics* **19**, 370-399.

[11] Herrero, M.-J. (1989) "The Nash Program: Non-convex Bargaining Problems." *Journal of Economic Theory* **49**, 266-277.

[12] Kalai, E. (1977) "Nonsymmetric Nash Solutions and Replications of 2-Person Bargaining." *International Journal of Game Theory* **18**, 155-162.

[13] Kaneko, M. (1980) "An Extension of the Nash Bargaining Problem and the Nash Social Welfare Function." *Theory and Decision* **12**, 135-148.

[14] Lehmann, E. and B. van der Linden (2007) "On the Optimality of Search Matching Equilibrium when Workers are Risk-Averse." *The Journal of Public Economic Theory* **9**, 867-884.

[15] Manning, A. (1987) "An Integration of Trade Union Models in a Sequential Bargaining Framework." *The Economic Journal* **97**, 121-139.

[16] Mas-Colell, A., M.D. Whinston and J.R. Green (1995) *Microeconomic Theory.* Oxford University Press: New York.

[17] Maschler, M., G. Owen, and B. Peleg (1988) "Path Leading to the Nash Set", in *The Shapley Value: Essays in Honor of L.S. Shapley* (A. Roth, Ed.), pp. 321-350, Cambridge Univ. Press, Cambridge, UK, 1988

[18] Nash, J.F. (1950) "The Bargaining Problem." *Econometrica* **18**, 155-162.

[19] Nash, J.F. (1953) "Two-Person Cooperative Games." *Econometrica* **21**, 128-140.

[20] Pissarides, C. (1985) "Short-run Equilibrium Dynamics of Unemployment, Vacancies, and Real Wages." *American Economic Review* **75**, 676-690.

[21] Pissarides, C. (2000) *Equilibrium Unemployment Theory,* second edition. MIT Press, Cambridge, Massachussets.

[22] Rogerson, R., R. Shimer, and R. Wright (2005) "Search-Theoretic Models of the Labor Market: A Survey." *Journal of Economic Literature* **43**, 959-988.

[23] Roth, A.E. (1977) "Individual Rationality and Nash's Solution to the Bargaining Problem." *Mathematics of Operations Research* **2**, 64-65.

[24] Rubinstein, A. (1982) "Perfect Equilibrium in a Bargaining Model." *Econometrica* **50**, 97-109.

[25] Rupert, P., M. Schindler, and R. Wright (2001) "Generalized Search-Theoretic Models of Monetary Exchange." *Journal of Monetary Economics* **48**, 605-622 .

[26] Serrano, R. and K-I. Shimomura (1998) "Beyond Nash Bargaining Theory: The Nash Set." *Journal of Economic Theory* **83**, 268-307 .

[27] Zhou, L. (1997) "The Nash Bargaining Theory with Non-Convex Problems." *Econometrica* **65**, 681-685.