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Solving Macroeconomic Models with Homogenous Technology and Logarithmic Preferences - A Note

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Abstract

In a recent published paper, Bethmann analyzed a stylised Robinson Crusoe economy, defining the state-like and control-like variables and then introducing the value-function-like function. He claims in the final section, that even if the model can be solved explicitly, but with complicated numerical analyses, he focuses instead on illustrating some dynamic programing techniques. In our opinion there are no complicated mathematical procedures to solve explicitly the model and we present here a closed-form solution of the model, only for the continuous case.

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1. Introduction

The concept of a Robinson Crusoe's economy is a simple framework to study economics, due to its ability to simplify the complexities of the real world. The implicit assumption is that the study of a one agent economy will provide useful insights into the functioning of the real world economy with many economic agents. This simple general economic system consists of a single household, and this one-person economy has many of the usual problems of any economy: production and consumption choices and allows us fully to present and to model the concept of efficient allocation. This is a centralized solution concept, since it treats the consumption and production decision in a single unified fashion and finds a production and consumption plan that maximizes Robinson's utility subject to the constraints of available resources and technology. This is exactly the problem analyzed by Bethmann (2013), in the spirit of the Lucas-Uzawa model, but under logarithmic preferences. A more general case was studied by Boucekkine and Ruiz-Tamarit (2008), and Chilarescu (2011). As he pointed out in his introduction section, similar models have been analyzed by Uzawa (1965), Lucas (1988), Caballe and Santos (1993), Mulligan and Sala-i-Martin (1993) and Xie (1994).

In his paper, in order to simplify the analysis, Bethmann considers the case of a closed economy populated by an arbitrary number of identical and infinitely-lived agents. The representative agent, Robinson Crusoe, enters every period with predetermined endowments of human and physical capital, denoted by $h = h(t)$ and $k = k(t)$, respectively. There are two sectors in the economy. Firms produce a single homogeneous good and a schooling sector produces education. Both sectors use constant returns to scale technologies in the reproducible factors. He also assume that the population is constant over time and normalised to unity, and therefore all variables can be interpreted as per capita quantities.

In order to arrive to our results, we preserve these assumptions and will analyze only the continuous case. The rest of the paper is structured as follows. In the second section we present the model, determine the first order conditions and give the relations that characterize the balanced growth path. In the third section we provide closed-form solutions for the original variables along the transitional dynamics path and give some important properties. In the last section, we present some numerical simulations and finally we present some conclusions.

2. The Model

In this section, we summarize the model proposed by Bethmann and derive the differential equations that describe the dynamics of the economy. The model is characterized by the well-known optimization problem.

Definition 1 *The set of paths* $\{k, h, c, u\}$ *is called an optimal solution if it solves the following optimization problem:*

$$V_0 = \max_{u,c} \int_0^\infty e^{-\rho t} ln\left[c(t)\right] dt, \tag{1}$$

subject to

$$\begin{cases} \dot{k}(t) = Ak(t)^\alpha \left[u(t)h(t)\right]^{1-\alpha} - c(t), \\ \dot{h}(t) = B[1 - u(t)]h(t), \\ k_0 = k(0), \ h_0 = h(0), \end{cases} \tag{2}$$

where $k_0 > 0$ *and* $h_0 > 0$ *are given,* α *is the elasticity of output with respect to physical capital,* ρ *is a positive discount factor, the efficiency parameters* $A > 0$ *and* $B > 0$ *represent the constant technological levels in the good sector and, respectively in the education sector,* k *is physical capital,* h *is human capital,* c *is the real consumption and* u *is the fraction of labor allocated to the goods production.*

The equations (2) give the resources constraints and initial values for the state variables k and h. To solve the problem (1) subject to (2), we define the Hamiltonian function:

$$H = ln(c) + \left[Ak^\alpha \left(uh\right)^{1-\alpha} - c\right] \lambda + B(1-u)h\mu.$$

The boundary conditions include initial values (k_0, h_0), and the transversality conditions:

$$\lim_{t\to\infty} e^{-\rho t}\lambda(t)k(t) = 0 \ \text{ and } \ \lim_{t\to\infty} e^{-\rho t}\mu(t)h(t) = 0.$$

In this model, there are two control variables, c and u, and two state variables, k and h. In an optimal program the control variables are chosen so as to maximize H. We note that along the optimal path, λ and μ are functions of t only.

A remark is absolutely necessary here. The above Hamiltonian is clearly non-concave in the control and state variables, due to the term $B(1-u)h$ and therefore, Mangasarian's conditions cannot be used in this case. Nevertheless Arrow's sufficiency theorem, that generalizes Mangasarian's result, still apply here. For more details we can cite, first of all the paper of Cysne (2006) and also the excellent book of Seierstad and Sydsaeter's (1987).

The necessary first order conditions for the pair (c, u) to be an optimal control are given by:

$$\begin{cases} \frac{\dot{k}}{k} = A\left(\frac{hu}{k}\right)^{1-\alpha} - \frac{c}{k}, \\ \frac{\dot{h}}{h} = B(1-u), \\ \frac{\dot{c}}{c} = -\rho + \alpha A\left(\frac{hu}{k}\right)^{1-\alpha}, \\ \frac{\dot{u}}{u} = \varphi - \frac{c}{k} + Bu,\ \varphi = \frac{B(1-\alpha)}{\alpha}, \\ \frac{\dot{\lambda}}{\lambda} = \rho - \alpha A\left(\frac{hu}{k}\right)^{1-\alpha}, \\ \frac{\dot{\mu}}{\mu} = \rho - B. \end{cases} \tag{3}$$

The system described above reaches the balanced growth path (BGP) if there exists t_* (possibly infinite), such that for all $t \geq t_*$, $r_u = 0$ and $r_k = r_c = r_h$, where r_x denotes the growth rate of variable x, x_* is its value at $t = t_*$ and x^* is its value for $t > t_*$. The following proposition gives a preliminary result that characterize the balanced growth path.

Proposition 1 *Let $\rho < B$. If for all $t \geq t_*$, $r_u = 0$, then the above system reaches the BGP and the following statements are valid*

i. $r_* = r_{k*} = r_{h*} = r_{c*} = B - \rho$,

ii. $u_* \in [0, 1]$ *and*
$$u_* = \frac{\rho}{B}, \tag{4}$$

iii.
$$\frac{c_*}{k_*} = \frac{\alpha\rho + B(1-\alpha)}{\alpha} = \rho + \varphi = \xi. \tag{5}$$

The proof follows immediately by direct computation.

From this proposition we deduce that it is not possible to compute the values of all variables at BGP. To find closed-form solutions to the optimal problem $(1)-(2)$, we need starting values for the control variables. These starting values obviously depend on the starting values of state variables and therefore should be computed as part of the closed-form solution. Without this assumption it is impossible to determine the values of variables (state and control) along the transitional dynamics and along the BGP. What we want to do next is to find a closed-form solution for the system described by (3), u_0 being determined as a function of the initial values of state variables.

3. The Closed-form Solution

As in the paper of Chilarescu (2011), we denote by

$$z(t) = \frac{h(t)u(t)}{k(t)},\ z_0 = \frac{h_0 u_0}{k_0}. \tag{6}$$

Obviously, z is a positive increasing function, in both variables, u ant t. (For more details concerning the properties of the function z, see the above cited paper). Differentiating (6) with respect to time we arrive to the following differential equation

$$\dot{z} = \left[\frac{B}{\alpha} - Az^{1-\alpha}\right] z. \tag{7}$$

A non-constant admissible solution of equation (7) is given by

$$z(t) = \frac{z_* z_0}{\left[\left(z_*^{1-\alpha} - z_0^{1-\alpha}\right) e^{-\varphi t} + z_0^{1-\alpha}\right]^{\frac{1}{1-\alpha}}},\ z_* = \left[\frac{B}{\alpha A}\right]^{\frac{1}{1-\alpha}}, \tag{8}$$

and this solution enable us to solve the system (3) and provide a closed-form solution. This result is given by the next theorem.

Theorem 1 *Let $B > \rho$. Then for all $t > 0$ the optimization problem $(1)-(2)$ has the following unique solution*

$$k(t) = \frac{h_0 u_0 e^{\xi t}}{z(t)} \frac{Q_* - Q(t)}{Q_*}\ e^{(B-\rho)t},$$

$$h(t) = \frac{h_0 e^{\xi t}}{\varphi} \frac{\varphi Q_* + Bu_0 \left[Q_* - Q1(t)\right] e^{-\varphi t} - Bu_0 \left[Q_* - Q(t)\right]}{Q_*}\ e^{(B-\rho)t},$$

$$c(t) = c(t) = \frac{h_0 u_0}{Q_*} z(t)^{-\alpha} e^{(B-\rho)t},$$

$$u(t) = \frac{\varphi u_0 [Q_* - Q(t)]}{[(\varphi + Bu_0) Q_* - Bu_0 Q1(t)] e^{-\varphi t} - Bu_0 [Q_* - Q(t)]},$$

where

$$Q(t) = \int_0^t z(s)^{1-\alpha} e^{-\xi s} ds,\ Q1(t) = \int_0^t z(s)^{1-\alpha} e^{-\rho s} ds,$$

$$Q_* = \lim_{t\to\infty} Q(t),\ Q1_* = \lim_{t\to\infty} Q1(t),\ Q1_* = \frac{\varphi + Bu_0}{Bu_0} Q_*,\ c_0 = k_0 Q_*^{-1} z_0^{1-\alpha}$$

and u_0 is the unique solution of the following equation

$$(\varphi + Bu_0) Q_*(u_0; k_0, h_0) - Bu_0 Q1_*(u_0; k_0, h_0) = 0. \tag{9}$$

Proof of Theorem 1. For proof that u_0 is the unique solution of *Eq.* (9), and other properties of the functions Q and $Q1$, see the paper of Chilarescu (2011). Substituting *Eq.* (8) into the third equation of the system (3) we arrive to the following differential equation

$$\frac{\dot{c}}{c} = -\rho + \frac{\alpha A z_*^{1-\alpha} z_0^{1-\alpha}}{\left(z_*^{1-\alpha} - z_0^{1-\alpha}\right) e^{-\varphi t} + z_0^{1-\alpha}},$$

whose solution is given by

$$c(t) = c_0 z_0^{\alpha} z^{-\alpha} e^{(B-\rho)t},\ \Rightarrow\ \lambda(t) = c_0^{-1} z_0^{-\alpha} z^{\alpha} e^{-(B-\rho)t},\ z = z(t).$$

The first equation of the system (3) can now be written

$$\dot{k} = A z^{1-\alpha} k - c_0 z_0^{\alpha} z^{-\alpha} e^{(B-\rho)t}.$$

After some algebraic manipulations, the solution for k is given by

$$k(t) = z_0 z^{-1} e^{\frac{B}{\alpha} t} \left[k_0 - c_0 z_0^{-1+\alpha} Q(t)\right].$$

Denoting $G(t) = k_0 - c_0 z_0^{-1+\alpha} Q(t)$, and since Q is a bounded positive function of time, $G(0) = k_0 > 0$, $\lim_{t\to\infty} G(t) = k_0 - c_0 z_0^{-1+\alpha} Q_*$, $Q_* = \lim_{t\to\infty} Q(t)$, and $\dot{G}(t) = -c_0 z_0^{-1+\alpha} z(t)^{1-\alpha} e^{-\xi t} < 0$, we deduce that G is a decreasing function of time. Since $k(t) > 0$ we deduce that $k_0 - c_0 z_0^{-1+\alpha} Q_* \geq 0$. Transversality condition for k requires that

$$c_0 = k_0 Q_*^{-1} z_0^{1-\alpha}, \tag{10}$$

and therefor we can write

$$k(t) = \frac{h_0 u_0}{z(t) Q_*} e^{\frac{B}{\alpha} t} [Q_* - Q(t)] = \frac{h_0 u_0 e^{\xi t}}{z(t)} \frac{Q_* - Q(t)}{Q_*} e^{(B-\rho)t}. \tag{11}$$

The solution for c follows immediately and is given by

$$c(t) = \frac{h_0 u_0}{Q_*} z(t)^{-\alpha} e^{(B-\rho)t}. \tag{12}$$

Passing to the limit in the above two relations and knowing that we get

$$k_* = \frac{h_0 u_0 e^{(B-\rho)t_*}}{\xi Q_* z_*^{\alpha}}, \text{ and } c_* = \frac{h_0 u_0 e^{(B-\rho)t_*}}{Q_* z_*^{\alpha}}.$$

The ratio $\frac{c}{k}$ is given by

$$\frac{c(t)}{k(t)} = \frac{z^{1-\alpha} e^{-\xi t}}{Q_* - Q(t)}, \tag{13}$$

and passing to the limit yields $\frac{c_*}{k_*} = \xi$. Combining the second and the last equations in (3) and considering the above result we arrive to the following differential equation

$$\frac{\dot{(hu)}}{hu} = \frac{B}{\alpha} - \frac{z^{1-\alpha} e^{-\xi t}}{Q_* - Q(t)},$$

whose solution is given by

$$h(t)u(t) = \frac{h_0 u_0}{Q_*} [Q_* - Q(t)] e^{\frac{B}{\alpha} t}. \tag{14}$$

Passing to the limit and knowing that $h^*(t) = h_* e^{(B-\rho)(t-t_*)}$, yields

$$\lim_{t \to \infty} h(t)u(t) = h_* u_* = \frac{h_0 u_0 e^{(B-\rho)t_*} z_*^{1-\alpha}}{\xi Q_*}.$$

Substituting $Eq.$ (13) into the fourth equation of the system (3) we arrive to the following differential equation

$$\frac{\dot{u}}{u} = \varphi - \frac{z^{1-\alpha} e^{-\xi t}}{Q_* - Q(t)} + Bu,$$

whose solution is given by

$$u(t) = \frac{\varphi u_0[Q_* - Q(t)]}{[(\varphi + Bu_0)\, Q_* - Bu_0 Q1(t)]\, e^{-\varphi t} - Bu_0[Q_* - Q(t)]}. \tag{15}$$

It is just an exercise to prove that $u(t) \in (0,1)$, the transversality condition for h holds and

$$u_* = \lim_{t\to\infty} u(t) = \frac{\rho}{B}, \Rightarrow \; h_* = \frac{Bh_0 u_0 e^{(B-\rho)t_*} z_*^{1-\alpha}}{\rho \xi Q_*}$$

and thus the proof is completed.

4. Conclusions and Numerical Simulations

In this paper we have provided closed-form solutions, for all the variables of the model analyzed by Bethmann. In this section we present the results of a numerical simulation procedure. The benchmark values for economy we consider are the following: $\alpha = 0.25$, $A = 1.05$, $B = 0.05$, $\rho = 0.04$, $h_0 = 10$, $k_0 = 80$. In order to find the transitional dynamics equations for the state and control variables we first determine the starting value for u_0. Under the above benchmark values, the equation (9) gives $u_0 = 0.76424$ and the equation (10) gives $c_0 = 14.532$. The corresponding steady-state equilibrium is given by: $u_* = 0.80$, $k_* = 99.776$, $h_* = 13.67$, $c_* = 18.95$. The transitional dynamics for these variables are presented at the end of this section. See the graphics from Fig. No. 1 to Fig. No. 4

Fig. No. 1 Transitional dynamics for k(t)

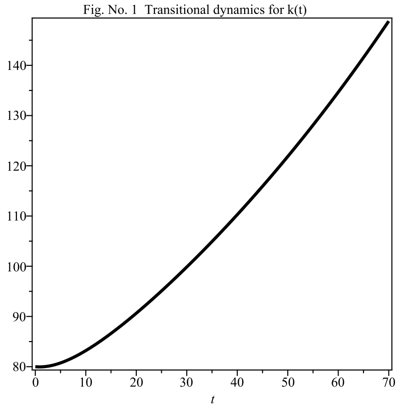
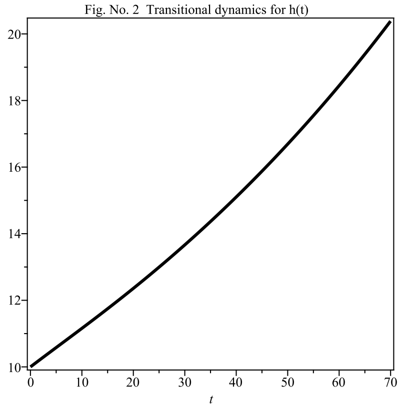
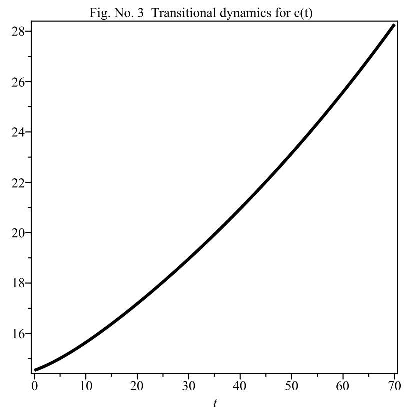
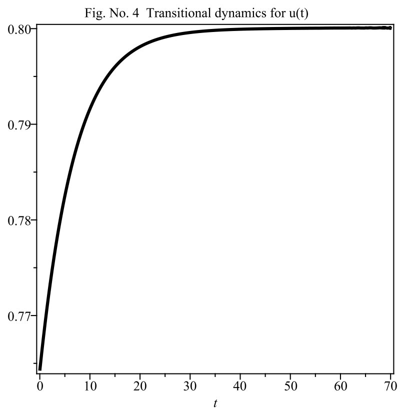


Fig. No. 2 Transitional dynamics for h(t)







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